GUDLAVALLERU ENGINEERING COLLEGE
(An Autonomous Institute with Permanent Affiliation to JNTUK, Kakinada)
Seshadri Rao Knowledge Village, Gudlavalleru – 521 356.

Department of Computer Science and Engineering

HANDOUT

on

DISCRETE MATHEMATICAL STRUCTURES

Vision :

To be a Centre of Excellence in computer science and engineering education and training to meet the challenging needs of the industry and society

Mission:

- To impart quality education through well-designed curriculum in tune with the growing software needs of the industry.
- To be a Centre of Excellence in computer science and engineering education and training to meet the challenging needs of the industry and society.
- To serve our students by inculcating in them problem solving, leadership, teamwork skills and the value of commitment to quality, ethical behavior & respect for others.
- To foster industry-academia relationship for mutual benefit and growth

Program Educational Objectives :

PEO1 : Identify, analyze, formulate and solve Computer Science and Engineering problems both independently and in a team environment by using the appropriate modern tools.

PEO2 : Manage software projects with significant technical, legal, ethical, social, environmental and economic considerations.

PEO3 : Demonstrate commitment and progress in lifelong learning, professional development, leadership and Communicate effectively with professional clients and the public

HANDOUT ON DISCRETE MATHEMATICAL STRUCTURES

Class & Sem. :II B.Tech – I Semester   Year   : 2018-19
Branch   : CSE   Credits   :   3

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1. Brief History and Scope of the Subject

The History of Foundations of Mathematics involve non classical logics and constructive mathematics. Mathematical Foundations of Computer Science is the study of mathematical structures that are fundamentally discrete rather than continuous. Research in Discrete Structures increased in the latter half of 20^{th} centenary partly due to development of digital computers, Which operate in Discrete steps and store data in discrete bits. Graph Theory is study of, Mathematical Structures used to model pair wise relations between objects from a certain collection. This course is useful in study and describing objects and problems in computer science such as computer algorithm, programming languages, Cryptography, Automated theorem proving and software development.

2. Pre-Requisites

- Mathematics background such as set theory, Permutations and Combinations.

3. Course Objectives:

To make the students

- know the structure of statements (and arguments) involving predicates.
- understand the applications of graph theory to various practical problems.
- know how to solve a recursive problem.

4. Course Outcomes:

Students will be able to

CO1: apply the concept of Mathematical logic in software development process.

CO2: use the concept of Pigeon hole principle to derive the $\Omega(n \log n)$ lower bound.

CO3: **a**pply the concepts of group theory in robotics, computer vision & computer graphics.

CO4: use the concepts of graph theory to provide solutions for routing applications in computer networks.

CO5: apply the recurrence relation for analyzing recursive algorithms.

5. Program Outcomes:

Graduates of the Computer Science and Engineering Program will have

a) an ability to apply knowledge of mathematics, science, and engineering
b) an ability to design and conduct experiments, as well as to analyze and interpret data
c) an ability to design a system, component, or process to meet desired needs within realistic constraints such as economic, environmental, social, political, ethical, health and safety, manufacturability, and sustainability
d) an ability to function on multidisciplinary teams
e) an ability to identify, formulate, and solve engineering problems
f) an understanding of professional and ethical responsibility
g) an ability to communicate effectively
h) the broad education necessary to understand the impact of engineering solutions in a global, economic, environmental, and societal context
i) a recognition of the need for, and an ability to engage in life-long learning,
j) a knowledge of contemporary issues
k) an ability to use the techniques, skills, and modern engineering tools necessary for engineering practice.

6. Mapping of Course Outcomes with Program Outcomes:

	a	b	c	d	e	f	g	h	i	j	k
CO1	2	3			2						2
CO2					3						2
CO3	3	3			3						3
CO4	3	3			3						3
CO5	2				2						1

7. Prescribed Text Books :

a) J.P.Trembley, R Manohar, Discrete Mathematical Structures with Applications to Computer Science, Tata McGraw Hill, New Delhi.

b) Mott, Kandel, Baker, Discrete Mathematics for Computer Scientists & Mathematicians, 2nd edition, PHI.
c) Rosen, Discrete Mathematics and its Application with combinatorics and graph theory: 7th editon, Tata McGraw Hill, New Delhi.

8. Reference Text Books

a) S.Santha, Discrete Mathematics, Cengage publications.
b) J K Sharma, Discrete Mathematics, 2nd edition, Macmillan Publications.

9. URLs and Other E-Learning Resources

So net CDs & IIT CDs on some of the topics are available in the digital library.

10. Digital Learning Materials:

- http://nptel.ac.in/courses/106106094
- http://nptel.ac.in/courses/106106094/40
- http://nptel.ac.in/courses/106106094/30
- http://nptel.ac.in/courses/106106094/32
- http://textofvideo. nptl.iitm.ac.in/106106094/lecl.pdf
- www.nptelvideos.in/2012/11/discrete-mathematical -structures.html

11. Lecture Schedule / Lesson Plan

Topic	No. of Periods	
	Theory	Tutorial
UNIT –1: Mathematical Logic :		
Propositional Calculus: Statements and Notations	1	2
Connectives	1	
Truth Tables	1	
Tautologies	1	2
Equivalence of Formulas	2	
Tautological Implications	1	
Theory of Inference for Statement Calculus	2	2
Consistency of Premises	1	
UNIT – 2: Relations & Functions		
Relations: Properties of Binary Relations	1	2
Equivalence	1	
Compatibility and Partial order relations	2	
Hasse Diagram	1	
Functions : Inverse	1	2
Composite and Recursive functions	2	
Pigeon hole principle and its application	1	
UNIT – 3: Algebraic Structures		
Algebraic Systems and Examples	1	2
general properties	1	
semi group, Monoid	1	
Groups	2	
Subgroups	2	2
Cyclic groups	2	
UNIT – 4: Graph Theory - I:		
Concepts of Graphs	1	2
Sub graphs, Multigraphs	2	
Matrix Representation of Graphs: Adjacency and incidence Matrices	2	2
Isomorphic Graphs	2	
UNIT – 5: Graph Theory - II:		
Paths and Circuits, Eulerian graph	2	2
Planar graphs	2	
Hamiltonian Graph	2	
Chromatic number of a graph	1	
UNIT – 6: Combinatorics and Recurrence Relation:		

Basics of Counting principles (sum rule and product rule)	1	2
Solving linear homogeneous recurrence Relations by substitution	1	
The Method of Characteristic Roots	2	2
Solving Inhomogeneous Recurrence Relations	2	
Total No. of Periods:	**48**	**24**

12. Seminar Topics

- Theory of Inference
- Graph isomorphism and applications
- Recurrence relations and applications

UNIT – I

Mathematical Logic

Objectives:

- To comprehend the structure of statements (and arguments) involving predicates and quantifiers

Syllabus:

Mathematical Logic: Propositional Calculus: Statements and Notations, Connectives, Truth Tables, Tautologies, Equivalence of Formulas, Tautological Implications, Theory of Inference for Statement Calculus, Consistency of Premises.

Sub Outcomes:

- Construct truth tables for different types of connectives.
- Identify the tautologies.
- Determine the equivalence formulas and tautological implications.

Learning Material

Statement:

A declarative sentence which is either true or false but not both is called a statement or proposition.

- Statements are generally denoted by either upper case or lower case letters.

Examples:

1. Bombay is the capital of Canada . (Statement)
2. Canada is a country. (Statement)
3. 10 + 100 = 110 . (Statement)
4. 3 + 3 = 4. (Statement)
5. x + 5 =8. (not a statement)
6. close the door. (not a statement)
7. what is your name? (not a statement)

Atomic statement/ Primitive statement :

A statement that cannot be broken down into more than one simpler statement is called atomic statement.

Compound statement:

A statement that can be broken down into simpler statements is called compound or molecular statement.

Propositional calculus:

The area of logic that deals with propositions is called propositional calculus or propositional logic.

Truth value:

- Truth value for true statement is T
- Truth value for false statement is F

Statement	Truth value
Bombay is the capital of Canada	F
Canada is a country	T
10 + 100 = 110	T
3 + 3 = 4	F
x + 5 =8	Not a statement We can't give truth value

Connectives:

The words or expressions which are used to construct compound statements from simpler statements are known as sentential connectives.

- And, or, if then, iff, not, so, because are sentential connectives.
- $+, -, \times, \div, \cup, \cap, \leq, \geq, <, >$ are mathematical connectives.

Different types of compound statements:

Type	Connective	Symbol	Notation	Read as
Conjunction	And	$\wedge$	$P \wedge Q$	P and Q
Disjunction	Or	$\vee$	$P \vee Q$	P or Q
Conditional	If then	$\rightarrow$	$P \rightarrow Q$	P implies Q i.e If P then Q
Bi-conditional	If and only if	$\leftrightarrow$	$P \leftrightarrow Q$	P double implies Q i.e. P if and only if Q
Negation	Not or No	$\sim$ or $\rceil$	$\sim P$	Negation P (or) Not P

Truth Table:

The table showing the truth values of a statement formula for each possible combination of the truth values of the compound statements is called the truth table of the formula.

Note: In general if there are n distinct components in a statement formula we need to consider 2^n possible combinations of truth values in order to construct the truth table.

Truth table rules:

P	Q	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$	$\sim P$
T	T	T	T	T	T	F
T	F	F	T	F	F	F
F	T	F	T	T	F	T
F	F	F	F	T	T	T

Example Problems:

Q. Using the statements R: Mark is rich. H: Mark is happy.

Denote the following statements in symbolic form.

a) Mark is poor but happy. **Ans:** $\backsim R \wedge H$
b) Mark is rich or unhappy. **Ans:** $R \vee \backsim H$
c) Mark is neither rich nor happy. **Ans:** $\backsim R \wedge \backsim H$
d) Mark is poor or he is both rich and unhappy. **Ans:** $\backsim R \vee (R \wedge \backsim H)$

Q. Represent the following statement in symbolic form.

" If either John takes Computer science or Merin takes Mathematics then Nishanth will take Biology."

Ans: Let us denote the statements as follows.
P : John takes Computer science
Q : Merin takes Mathematics
R : Nishanth takes Biology
Then given statement can be written as $(P \vee Q) \longrightarrow R$.

Q. How can the following statement be translated into a logical expression.

" You can access the internet from campus only if you are a Computer science major student or you are not a freshman." (For student)

Q. Construct the truth tables for the following statement formula.

1. $\backsim(\backsim P \vee \backsim Q)$

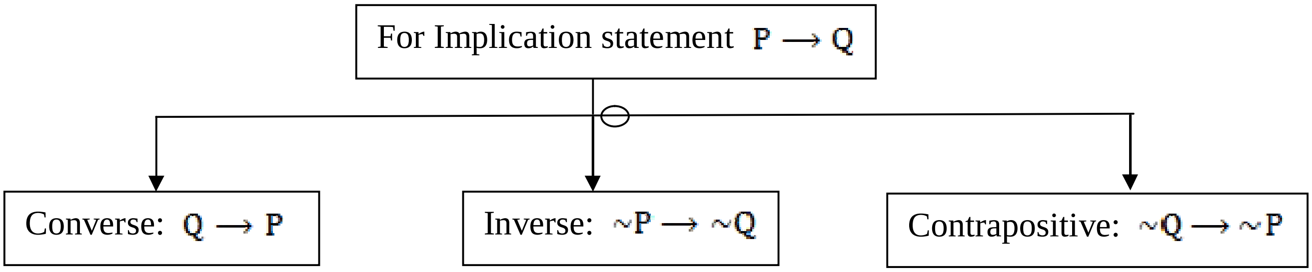
P	Q	$\backsim P$	$\backsim Q$	$\backsim P \vee \backsim Q$	$\backsim(\backsim P \vee \backsim Q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

2. $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

P	Q	R	$\neg P$	$\neg Q$	$\neg Q \wedge R$	$\neg P \wedge (\neg Q \wedge R)$ A	$Q \wedge R$ B	$P \wedge R$ B	$A \vee B$	$A \vee B \vee C$
T	T	T	F	F	F	F	T	T	T	T
T	T	F	F	F	F	F	F	F	F	F
T	F	T	F	T	T	F	F	T	F	T
T	F	F	F	T	F	F	F	F	F	F
F	T	T	T	F	F	F	T	F	T	T
F	T	F	T	F	F	F	F	F	F	F
F	F	T	T	T	T	T	F	F	T	T
F	F	F	T	T	F	F	F	F	F	F

3. $[Q \wedge (P \longrightarrow Q)] \rightarrow P$ (For student)

4. $\sim(P \wedge Q) \leftrightarrow \sim P \vee \sim Q$ (For student)



Note: Converse of inverse of an implication is a contrapositive.

Inverse of connverse of an implication is a contrapositive.

Conditional ($P \longrightarrow Q$)	If I am sleeping, then I am breathing
Converse ($Q \longrightarrow P$)	If I am breathing, then I am sleeping
Inverse ($\sim P \longrightarrow \sim Q$)	If I am not sleeping, then I am not breathing
Contrapositive ($\sim Q \longrightarrow \sim P$)	If I am not breathing, then I am not sleeping

Q. What are the converse, inverse and contrapositive of the implication

"If I get good rank in EAMCET then I will choose CSE. "

Ans: Let us take the statements as follows.

P: I get good rank in EAMCET

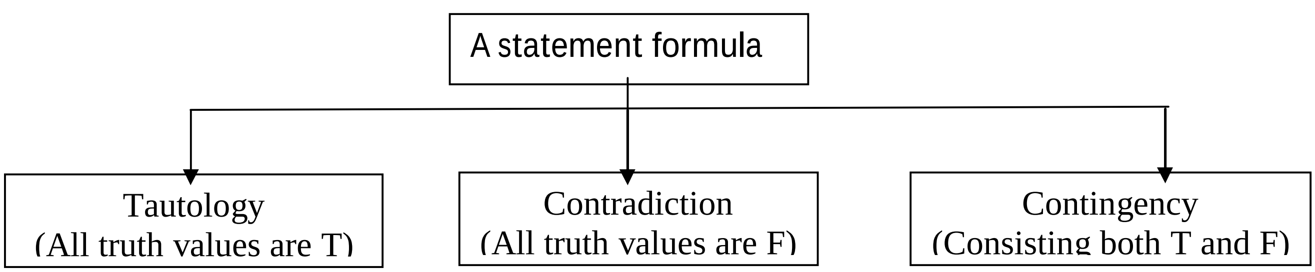
Q: I will choose CSE

Converse: If I choose CSE, then I got good rank in EAMCET.

Inverse : If I not get good rank in EAMCET , then I will not choose CSE.

Contrapositive: If I will not choose CSE, then I did not get good rank in EAMCET.

Q. What are the inverse, converse, and contra positive of the implication "If today is a holiday, then I will go for a movie " (For student)



Tautology:

A statement formula that is always true, irrespective of the truth values of the propositions that occur in it, is called a tautology. This is also called as universally valid formula or a logical truth.

Contradiction:

A statement formula that is always false, irrespective of the truth values of the propositions that occur in it, is called contradiction.

Contingency:

A proposition that is neither a tautology nor a contradiction is called a contingency.

Note: 1. The negation of contradiction is tautology.

2. The conjunction of two tautologies is also a tautology.

Q. Indentify that $[(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))]$ is a tautology.

p	**Q**	**r**	$p \rightarrow q$ **A**	$q \rightarrow r$ **B**	$p \rightarrow r$ **C**	$p \rightarrow B$ **D**	$A \rightarrow C$ **E**	$D \rightarrow E$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Hence the given statement is Tautology.

Q. Show that $((\neg q \wedge p) \wedge q)$ is a contradiction.

P	**q**	$\neg q$	$\neg q \wedge p$	$((\neg q \wedge p) \wedge q)$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

Hence the given statement is Contradiction.

Equivalence formulas:

The two propositions A and B are said to be logically equivalent if $A \leftrightarrow B$ is a tautology.

And written as $A \Leftrightarrow B$ and read as A is equivalent to B.

Q. show that $(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$

P	**Q**	$\sim P$	$P \rightarrow Q$	$\sim P \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Hence $(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$

Q. Show that $((P \wedge \neg P) \vee Q) \Leftrightarrow Q$

P	**Q**	$\neg P$	$P \wedge \neg P$	$(P \wedge \neg P) \vee Q$
T	T	F	F	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	F

From the above truth table,

Hence $((P \wedge \neg P) \vee Q) \Leftrightarrow Q$

Equivalence Rules :

The logical equivalences below are important equivalences that should be memorized.

Idempotent Laws: $p \vee p \Leftrightarrow p$

$p \wedge p \Leftrightarrow p$

Commutative Laws: $p \vee q \Leftrightarrow q \vee p$

$p \wedge q \Leftrightarrow q \wedge p$

Associative Laws: $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

Distributive Laws: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Identity Laws: $p \wedge T \Leftrightarrow p$

$p \vee F \Leftrightarrow p$

Domination Laws: $p \vee T \Leftrightarrow T$

$p \wedge F \Leftrightarrow F$

De Morgan's Laws: $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

Absorption Laws: $p \wedge (p \vee q) \Leftrightarrow p$

$p \vee (p \wedge q) \Leftrightarrow p$

Negation Laws: $p \vee \neg p \Leftrightarrow T$

$p \wedge \neg p \Leftrightarrow F$

Double Negation Law: $\neg(\neg p) \Leftrightarrow p$.

Tautological Implications:

A statement A is said to be tautologically imply to a statement B iff $A \rightarrow B$ is a tautology. And it is denoted by $A \Rightarrow B$.

Note: 1.If a statement formula is equivalent to tautology then it must be a tautology.

2.If a formula is implied by a tautology then it must be tautology.

Other connectives:

Type	Symbol	Definition
Exclusive OR i.e. XOR	$\bar{\vee}$	$(P \bar{\vee} Q) \Leftrightarrow \urcorner(P \leftrightarrow Q)$
NAND	$\uparrow$	$P \uparrow Q \Leftrightarrow \urcorner(P \wedge Q)$
NOR	$\downarrow$	$P \downarrow Q \Leftrightarrow \urcorner(P \vee Q)$

Truth table:

P	Q	$P \leftrightarrow Q$	$P \wedge Q$	$P \vee Q$	XOR $P \bar{\vee} Q$	NAND $P \uparrow Q$	NOR $P \downarrow Q$
T	T	T	T	T	F	F	F
T	F	F	F	T	T	T	F
F	T	F	F	T	T	T	F
F	F	T	F	F	F	T	T

Theory of Inference

The main function of logic is to provide rules of inference or principles of reasoning. The theory associated with such rules is known as inference theory.

Premise: Premise is an axiom or believed to be true either from experience or from faith.

Valid conclusion and Valid argument:

Any conclusion which is arrived by the set of rules or premises is called a valid conclusion and the argument is called a valid argument.

Validity using truth tables:

Q. Determine whether the conclusion C follows logically from the premises H_1 and H_2 in the following cases.

1. $H_1: P \rightarrow Q$ $H_2: \neg(P \wedge Q)$ $C: \neg P$

Sol: We have to construct the following truth tabe.

P	**Q**	H_1 $P \rightarrow Q$	$P \wedge Q$	H_2 $\neg(P \wedge Q)$	C $\neg P$
T	T	T	T	F	F
T	F	F	F	T	F
F	T	**T**	F	**T**	**T**
F	F	**T**	F	**T**	**T**

Here H_1 and H_2 are the true in the third and fourth rows and the conclusion C is also T in these two rows.

Thus . $H_1: P \rightarrow Q,\ H_2: \neg(P \wedge Q) \Rightarrow C: \neg P$

Hence the conclusion is valid.

2. $H_1: P \rightarrow Q$ $H_2: Q$ $C: P$

Sol: We have to construct the following truth tabe.

(C) P	**(H_2) Q**	(H_1) $P \rightarrow Q$
T	**T**	**T**
T	F	F
F	**T**	**T**
F	F	T

Here H_1 and H_2 are true in the 1^{st} and 3^{rd} rows but conclusion C is true only in 1^{st} row false in 3^{rd} row.

Hence the conclusion C is not valid.

Rules of inferences:

Rule P: A premise may be introduced at any point in the derivation.

Rule T:A formula S may be introduced in a derivation if S is tautologically implied by one or more of the preceding formulas in the derivation.

Rule CP: If we can derive S from R and a set of premises, then we can derive $R \rightarrow S$ from a set of premises alone.

Q. Apply theory of inference to check R is valid inference from the premises
$P \rightarrow Q, Q \rightarrow R, P$

Sol:

{ 1 }	(1). P	Rule P
{ 2 }	(2). $P \rightarrow Q$	Rule P
{ 1,2 }	(3).Q	Rule T on (1),(2) "$P. P \rightarrow Q \Rightarrow Q$"
{ 4}	(4).$Q \rightarrow R$	Rule P
{ 1,2,4 }	(5).R	Rule T on (3),(4) "$P. P \rightarrow Q \Rightarrow Q$"

Hence R is valid inference

Q. Show that $S \vee R$ is automatically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$

{ 1 }	(1). $P \vee Q$	Rule P
{ 1 }	(2). $\neg P \rightarrow Q$	Rule T on (1) "$P \rightarrow Q \Leftrightarrow \neg P \vee Q$"
{ 3 }	(3).$Q \rightarrow S$	Rule P
{ 1,3}	(4).$\neg P \rightarrow S$	Rule T on (2),(3) "$P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$"
{ 1,3 }	(5). $\neg S \rightarrow P$	Rule T on (4) "$P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$"
{ 6 }	(6). $P \rightarrow R$	Rule P
{ 1,3,6 }	(7). $\neg S \rightarrow R$	Rule T on (5),(6) "$P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$"
{ 1,3,6 }	(8). $S \vee R$	Rule T on (7) "$P \rightarrow Q \Leftrightarrow \neg P \vee Q$"

Hence proved

Q. Show that $R \vee S$ follows logically from the premises

$C \vee D, C \vee D \rightarrow \neg H, \neg H \rightarrow (A \wedge \neg B)$ and $(A \wedge \neg B) \rightarrow R \vee S$ (For student)

Consistency of premises: A set of formulas H_1, H_2 ……..H_n is said to be consistent if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in H_1, H_2, …….H_m. A set of formulas H_1, H_2……..H_m is said to be inconsistent if their conjunction implies a contradiction i. e. $H_1 \wedge H_2 \wedge ....H_m \Rightarrow R \wedge \neg R$ where R is any formula.

Q. Show that the set of premises

$P \rightarrow Q, P \rightarrow R, Q \rightarrow \neg R, P$ are in consistent

Sol:

1)	$P \rightarrow Q$	P		
2)	$Q \rightarrow \neg R$	P		
3)	$P \rightarrow \neg R$	T	1,2	$P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$
4)	P	P		
5)	$\neg R$	T	3,4	$P \rightarrow Q, P \Rightarrow Q$
6)	$P \rightarrow R$ P			
7)	$\neg P$	T	5,6	$P \rightarrow Q, \neg Q \Rightarrow \neg P$
8)	P	P		
9)	$P \wedge \neg P$	T	7,8	$P, Q \Rightarrow P \wedge Q$

The set of premises are inconsistent

UNIT-I
Assignment-Cum-Tutorial Questions
SECTION-A

Objective Questions

1) Which of the following is a statement. [ ]

a) how old are you ? b) Jaipur is in Andhra Pradesh c) where are you ? d) god bless you.

2) If p and q are two statements then the converse of $\neg q \rightarrow \neg p$: _________________

3) The inverse of $(p \wedge q) \rightarrow (\neg q \wedge \neg p)$ is ______________

4) The negation of the statement 'there are 7 days in a week ' is _____________

5) What is the truth value of the statement 'If Charminar is in Hyderabad then 5*3=8'. [T / F]

6) If the truth value of q is T then the truth value of $(q \vee r) \wedge q$ is ____ .

7) The truth value of 2+6=9 if and only if 9+6=10 is ______ .

8) The converse of the statement "If there is a flood then the crop will be destroyed" is ____ .

9) Symbolic form of the statement 'If I do not have car or I do not wear good dress then I am not a Millionaire' is _________________ .

10) P and Q are two propositions. Which of the following logical expressions are equivalent?

I. $P \vee \sim Q$

II. $\sim(\sim P \wedge Q)$

III. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

IV. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$ [ ]

a) Only I and II b) Only I, II and III c) Only I, II and IV d) All of I, II, III & IV

11) Consider the following propositional statements:

$$P_1 : ((A \wedge B) \rightarrow C) \equiv ((A \rightarrow C) \wedge (B \rightarrow C))$$

$$P_2 : ((A \vee B) \rightarrow C) \equiv ((A \rightarrow C) \vee (B \rightarrow C))$$

Which one of the following is true? [ ]

a) P_1 is a True, but not P_2

b) P_2 is a True, but not P_1

c) P_1 and P_2 are both True

d) Both P_1 and P_2 are not True

12) Consider the following statements

P: Good mobile phones are not cheap

Q : Cheap mobile phones are not good.

L : P implies Q.   M : Q implies P .   N : P is equivalent to Q

Which of the following about L , M and N is correct .   [ ]

a) only L is true   b) only M is correct

c) only N is true   d) L, M and N are true.

13) The negation of the statement '$(P \vee Q \vee R)$' is   [ ]

a) $\sim P \wedge \sim Q \wedge R$ b) $\sim P \wedge \sim Q \wedge \sim R$ c) $\sim P \wedge \sim Q \wedge R$ d) $\sim P \vee \sim Q \vee \sim R$

14) Which of the following is a tautology ?   [ ]

a) $\neg p \Rightarrow (p \wedge q)$ b) $(((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p)) \Rightarrow r$

c) $p \Rightarrow p \vee q$ d) $p \wedge q$

15) Which of the following is a contingency?   [ ]

a) $(p \wedge q) \Rightarrow (p \vee q)$ b) $p \vee q \Rightarrow (p \wedge q)$

c) $p \vee \neg p$ d) $p \wedge q \Rightarrow p$

SECTION-B

SUBJECTIVE QUESTIONS

1) Let p , q and r be the propositions . P: you have the free.
Q: you miss the final examination.
R: you pass the course.

Write the following proposition into statement form.

i) P → Q   ii) □ P → R   iii) Q → □ R   iv) P V Q V R   v) (P → □R) V (Q → □R)

2) Construct a truth table for each of the following compound statements.

i) (p → q) □ (□ p → q)   ii) p → (□ q v r)

3) Construct the truth table for the given statement: (P→(Q→R))→((P→Q)→(P→R)).

4) Construct the truth table for $p \wedge (p \rightarrow q)$

5) Construct the truth table for $[(PVQ) \wedge \sim R] \leftrightarrow Q$.

6) Show that $p \rightarrow q \Leftrightarrow \sim p \ V \ q$.

7) Show that $(P \rightarrow (Q \rightarrow R)) \Leftrightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$.

8) Use truth table to verify the following logical equivalence $p \rightarrow (q \wedge r) \Leftrightarrow (p \rightarrow q) \wedge (p \rightarrow r)$

9) Establish the validity of the argument $p \rightarrow q, \ q \rightarrow r, p \ \Rightarrow \ r$.

10) Show that R V S follows logically form the premises C v D , (C v D) → ~ H, ~H → (A ∧~B) and (A ∧~B) → (R v S).

11) Determine the validity of the following argument : " my father praises me only if I can be proud of myself either I do well in sports or I cann't be proud of myself. If I study hard, then I cann't do well in sports. Therefore, if father praises me then I do not study well."

12) Show that the following set of premises is inconsistent :
" if the contract is valid then john is liable for penalty. "If john is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the contract is valid and the bank will loan him money."

13) Prove that the following argument is valid.
If Rochelle gets the supervisor's position and works hard, then she'll get a raise. If she gets the raise, then she'll buy a new car. She has not purchased a new car. Therefore either Rochelle did not get the supervisor's position or she did not work hard.

SECTION-C

QUESTIONS AT THE LEVEL OF GATE

1.Which one of the following is NOT equivalent to $p \leftrightarrow q$?
(A) (┐p □ q) □ (p □ ┐q) (B) (┐p □ q) □ ($q \rightarrow p$)
(C) (┐p □ q) □ (p □ ┐q) (D) (┐p □ ┐q) □ (p □ q)

(GATE2015)

2.Let a, b, c, d be propositions. Assume that the equivalences a ↔ (b V-b) and b ↔ c hold. Then the truth value of the formula (a ∧ b) → (a ∧ c) ∨ d) is always [ ]
(A**)** True (B**)** False (C**)** Same as the truth value of b (D**)** Same as the truth value of d **(GATE 2000)**

3. P and Q are two propositions. Which of the following logical expressions are equivalent?

I. $P \vee \sim Q$
II. $\sim(\sim P \wedge Q)$
III. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$
IV. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$

a)Only I and II b)Only I, II and III

c)Only I, II and IV d)All of I, II, III and IV (**GATE 2008)**

4.Which one of the following Boolean expressions is NOT a

(A) $((a \longrightarrow b) \wedge (b \longrightarrow c)) \longrightarrow (a \longrightarrow c)$

(B) $(a \leftrightarrow c) \longrightarrow (\backsim b \longrightarrow (a \wedge c))$

(C) $(a \wedge b \wedge c) \longrightarrow (c \vee a)$

(D) $a \longrightarrow (b \longrightarrow a)$

tautology?

a) A b) B c) C d) D **(GATE 2014)**

5. Let P, Q and R be three atomic prepositional assertions. Let X denote (P v Q) → R and Y denote (P → R) v (Q → R). Which one of the following is a tautology?

a) X ≡ Y b) X → Y c) Y → X d) ¬ Y → X **(GATE-CS-2005)**

@@@

UNIT – II

Relations & Functions

Objectives:

- To use the concept of pigeonhole principle to derive the Ω (n log n) lower bound and to draw the Hasse diagram.

Syllabus:

Relations: Properties of Binary Relations, Equivalence, Compatibility and Partial order relations, Hasse Diagram
Functions: Inverse, Composite and Recursive functions, Pigeon hole principle and its application.

Sub Outcomes:

- Classify various types of binary relations .
- Draw the Hasse diagram for the given relation.
- Evaluate the inverse of a function.
- Use the composite operation to find the primitive recursion of the given function.
- Use the concept of pigeonhole principle to derive the Ω (n log n) lower bound.

Learning Material

Relations:

Let A and B be two sets. A binary relation or, simply, relation from A to B is a subset of A X B.

Suppose R is a relation from A to B, then R is a set of ordered pairs where each first element comes from A and each second element comes from B. For each pair $a \in A$ and $b \in B$, exactly one of the following is true:

(i) (a, b) $\in$ R: we then say that "a is R-related to b", written a R b.
(ii) (a , b) $\notin$ R: we then say that " a is not R-related to b" ,

If R is a relation from a set A to itself, if R is a subset of A^2 = A X A, then we say that R is a relation on A.

The domain of a relation R is the set of all first elements of the ordered pairs which belongs to R, and the range of R is the set of second elements.

Problem 1: Let A = { 1 , 2, 3 } and B ={ x , y , z }and let R = { (1 , y) ,(1 , z) , (3 , y) }. Then R is a relation from A to B since R is a subset of A X B.

Composition of Relations:

Let A, B and C be sets and let R be a relation from A to B and let S be a relation from B to C. R is a subset of A X B and S is a subset of B X C. Then R and S gives relation from A to C, which is denoted by

R o S = { (a ,c) : there exists b ∈B for which (a , b) ∈ R and (b , c) ∈ S }

Problem : Let A = { 1 , 2, 3 , 4 } , B = { a , b , c , d } , C = { x , y , z } and let

R = { (1 , a) ,(2 ,d) , (3 ,a), (3 ,b) ,(3 ,d) } and S = {(b , x) ,(b , z) , (c , y) ,(d , z)} then

R o S = {(2 , z) ,(3 ,x) , (3 , z)}.

Reflexive relation:

A relation R on a set A is reflexive if aRa for every a ∈A, if (a, a) ∈ R for every a ∈A.

Ex: Consider the following relations on the set A = {1, 2, 3}; then

R = {(1 , 2) ,(1 ,3) , (3 , 2)} is not a reflexive relation.

R = { (1 , 1) ,(2 , 2) , (3 ,3)} is reflexive relation.

Symmetric relation:

A relation R on a set A is symmetric if whenever aRb then bRa .i.e., (a, b)∈R , then (b , a) ∈ R

- If a = b in above R, then R is called anti-symmetric

Remark:

The properties of being symmetric and being anti symmetric are not negatives of each other.

Ex: 1) The relation R = {(1, 3), (3 , 1) ,(2 , 3) } is neither symmetric nor anti symmetric

2)Consider the relation R = {(1, 3), (3, 1)} is both symmetric and anti symmetric.

Equivalence Relation:

A relation R in a set X is called on equivalence relation if it is reflexive, symmetric and transitive

Ex: x = {1,2………7} and R = {(x,y) / x-y is divisible by 3} is an equivalence relation.

Compatibility Relations:

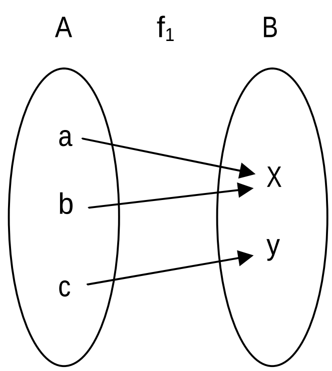
A relation R in X is said to be a compatibility relation if it is reflexive and symmetric. and is given byR = { (x , y)/x, y ∈ X x R y if x and y contain some common letter}

Partial Order Relation: A binary relation R in a set P is called a partial order relation or a partial ordering in P iff R is reflexive, antisymmetric, and transitive. If $\leq$ is a partial ordering on P, then the ordered pair $(P, \leq)$ is called a partially ordered set or a poset.

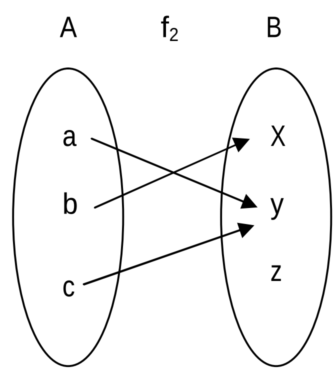
EX:- Let R be the set of real numbers. The relation "less than or equal to," is a partial ordering on R.

Function:

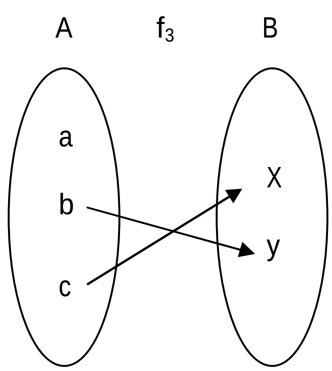
A relation $f: A \longrightarrow B$ is said to be a function if every element in A has unique image in B.



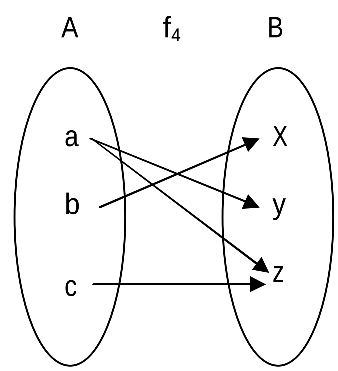
f_1 is a function



f_2 is a function



f_3 is not a function



f_4 is not a function

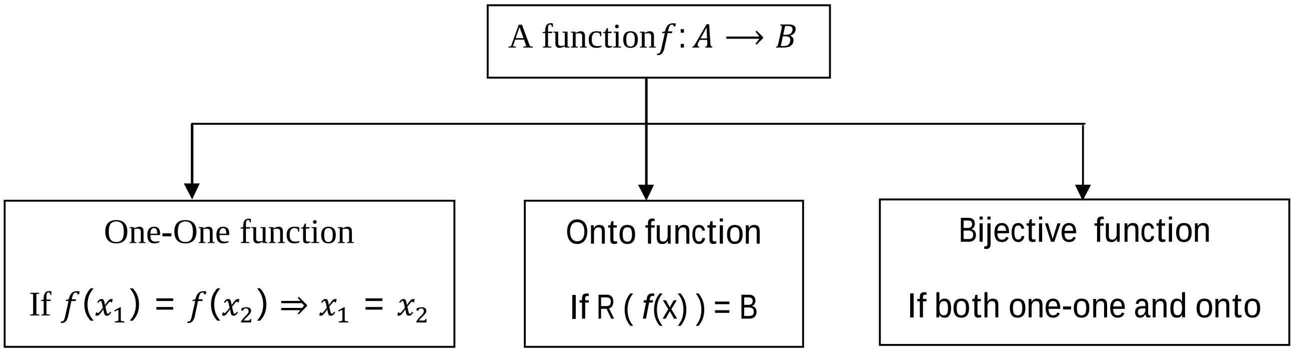
Domain and Co-domain:

In a function $f: A \longrightarrow B$

- A is called Domain and
- B is called Co-domain.

Range: The set of all images with respect to a function$f$ is range of f.

Types of functions:



1. **One-One function (or) Injective function:**
   A function f(x) is said to be one-one function
   - $iff(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

2. **Onto function (or) Surjective function:**
   A function f (x) is said to be Onto function
   - if range of f (x) = co-domain of f (x).
   - Otherwise into function.

3. **Bijective function:**
   A function f (x) is said to be bijective if is both One-One and Onto.

Q.State which of the following are injections or bijections from R into R, where R is the set of all real numbers

i) $f(x) = -2x$ ii)$f(x) = x^2 - 1$

Sol:

i) Given function $f(x) = -2x$

Injection or One-one:

Let $f(x_1) = f(x_2)$

$-2x_1 = -2x_2$

$x_1 = x_2$

Hence *f(x)* is one-one function

Surjection or onto:

Let $f(x) = y$

$-2x = y \Rightarrow x = -\frac{y}{2}$

$\therefore x \in R for all y \in R$

Hence *f(x)* is Onto

Thus *f(x)* is One – one and Onto function.

Hence *f(x)* is Bijection function.

ii) Given function $f(x) = x^2 - 1$

Injection or One-one:

Let $f(x_1) = f(x_2)$

$x_1^2 - 1 = x_2^2 - 1$

$x_1^2 = x_2^2$

$x_1 = \pm x_2$

Hence *f(x)* is not one-one function

Surjection or onto:

Let $f(x) = y$

$x^2 - 1 = y$

$x^2 = y + 1$

$x = \pm\sqrt{(y} + 1)$

Hence *f(x)* is not Onto

Thus *f(x)* is not One – one and not Onto function.

Hence *f(x)* is not Bijection function.

Inverse Function:

Let $f: A \longrightarrow B$ be a function. If $f^{-1}: B \longrightarrow A$ is also a function then

- f is said to be invertible and
- f^{-1} is an inverse function of f.

Note: A function $f: A \longrightarrow B$ invertible $\Leftrightarrow f$ is one-one and onto i.e.,bijection.

Q. Find the inverse of the function $f(x) = 4e^{6x+2}$

Sol: Given $f(x) = 4e^{6x+2}$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$$4e^{6x+2} = y$$

$$6x + 2 = \ln\left(\frac{y}{4}\right)$$

$$x = \frac{1}{6}[\ln\left(\frac{y}{4}\right) - 2]$$

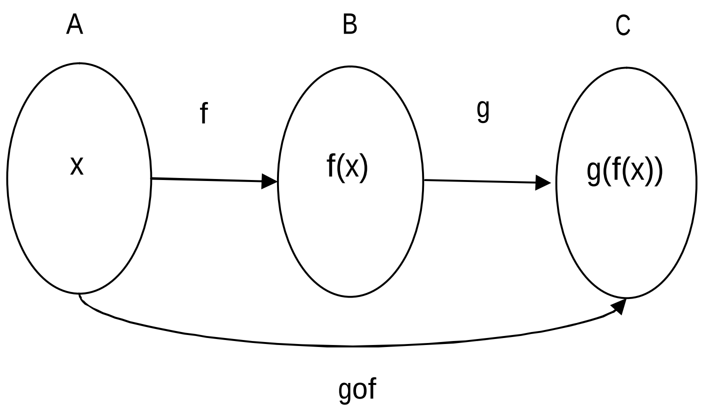
$$f^{-1}(y) = x = \frac{1}{6}[\ln\left(\frac{y}{4}\right) - 2]$$

$$\therefore f^{-1}(x) = \frac{1}{6}[\ln\left(\frac{x}{4}\right) - 2]$$

Composition of functions:

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be two functions. The composition of f and g,

- Denoted by gof,
- Is a function from A to C,
- Defined as $(gof)(x) = g(f(x)), for all x \in A$.

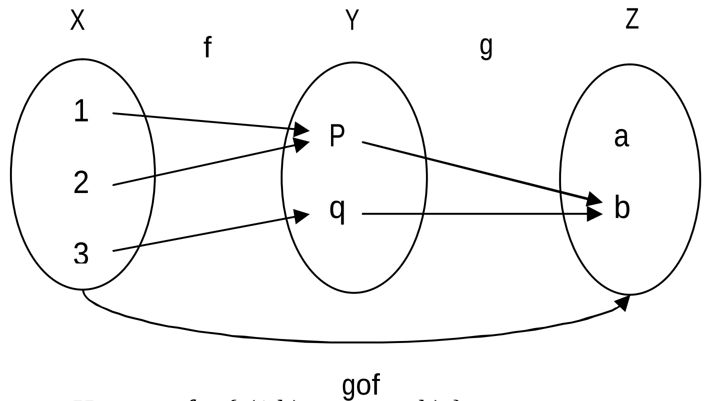


To find Compositions:

- $(fog)(x) = f(g(x))$
- $(gof)(x) = g(f(x))$
- $f^2(x) = (fof)(x) = f(f(x))$
- $(fogoh)(x) = (fo(goh))(x) = ((fog)oh)(x) = f(g(h(x)))$

Q. Let X= { 1, 2, 3 }, Y = { p, q } and Z = { a, b }. Also let $f: X \longrightarrow Y$ be f = { (1, p), (2, p), (3,q) } and $g: Y \longrightarrow Z$ be given by g= { (p, b), (q,b) }. Find gof.

Sol: Given $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ then $gof: X \longrightarrow Z$



Hence *gof* = { (1,b), ([illegible] ,b) }

Q. Let $f(x) = x + 2, g(x) = x - 2, and h(x) = 3x for x \in R$, where R is the set of real numbers. Find *gof; fog; fof; gog; foh; hog; hof; and fogoh.*

Sol: Given $f(x) = x + 2, g(x) = x - 2, and h(x) = 3x$

$$gof(x) = g(f(x)) = g(x+2) = x + 2 - 2 = x$$

$$fog(x) = f(g(x)) = f(x-2) = x - 2 + 2 = x$$

$$fof(x) = f(f(x)) = f(x+2) = x + 2 + 2 = x + 4$$

$$gog(x) = g(g(x)) = g(x-2) = x - 2 - 2 = x - 4$$

$$foh(x) = f(h(x)) = f(3x) = 3x + 2$$

$$hog(x) = h(g(x)) = h(x-2) = 3(x-2) = 3x - 6$$

$$hof(x) = h(f(x)) = h(x+2) = 3(x+2) = 3x + 6$$

$$fogoh(x) = f\left(g(h(x))\right) = f(g(3x)) = f(3x - 2) = 3x - 2 + 2 = 3x$$

Recursive Function:

- Recursion is a technique of defining a function, a set or an algorithm in terms of itself.
- First specify the value of the function at zero and give a rule for finding its value at an integer from its values at smaller integers. This is called a recursive.

Initial functions:

1. **Zero function Z :** Z(x) = 0
2. **Successor function S:**$S(x) = x + 1$
3. **Projection function U_i^n** : $U_i^n(x_1, x_2, \dots\dots, x_i, \dots\dots x_n) = x_i$ (generalized identity function)

Primitive Recursive function:

A function *f*(x) is said to be primitive recursive function if it satisfies

- $f(0) = k$
- $f(x+1)$can be represented in terms of successor function and / or projection function includes x and / or *f* (x).

Q. Show that $f(x, y) = x + 2y$ is primitive recursive function?

Sol: Given function $f(x, y) = x + 2y$

1. $f(x, 0) = x$
2. $f(x, y + 1) = x + 2(y + 1)$

$$= x + 2y + 2$$
$$= f(x, y) + 2$$
$$= S\left(S\big(f(x, y)\big)\right)$$
$$= S(S\left(U_3^3\big(x, y, f(x, y)\big)\right))$$

Hence *f(x, y)* is primitive recursive function.

Pigeonhole Principle:

- If n + 1 objects (pigeons) are put into n boxes (pigeonholes), then at least one box contains two or more objects.

Generalization:

- If N pigeons are placed in K pigeonholes, where N> K, then at least one pigeonhole must contains $\left\lfloor \frac{N-1}{K} \right\rfloor + 1$pigeons. Here $\lfloor . \rfloor$ denotes the floor function.

Problems:

Q: Prove that among 13 people, there are two born in the same month.

Sol: There are n = 12 months (‘boxes’), but we have n+1 = 13 people (‘objects’). Therefore two people were born in the same month.

Q: How many persons must be chosen in order that at least five of them will have birthdays in the same calendar month?

Sol: Let n be the required no.of persons. Since the number of months over which the birthdays are distributed is 12, the least no.of persons who have their birthdays is 5.

By the generalized pigeonhole principle $\left\lfloor \frac{N-1}{K} \right\rfloor + 1$ =5 $\Rightarrow$ n = 5.

Q: Prove that if 30 dictionaries in a library contain a total of 61,327 pages, then at least one of the dictionaries must have at least 2045 pages. (For Student)

UNIT-II
Assignment-Cum-Tutorial Questions
SECTION-A

Objective Questions

1. Let R = { (1,1) , (2,2) ,(3,3) } be a relation in the set A = { 1,2,3} then R is [ ]
   a) Symmetric b) Anti symmetric c) Both a and b d) Neither a Nor b
2. If A = { 1,3,5,7} and B ={ 2,4,5,6,7} then which of the following set of ordered points represents a function from A to B [ ]
   a) {(1,2),(5,6) , (3,4)} b) {(1,2),(1,6),(3,4),(5,7),(7,6)}
   c) {(1,2),(5,6) , (3,4),(7,7)} d) {(1,2),(5,6) , (3,4),(6,7)}
3. Let A = {1,2,3} and R = {(1,1),(1,2), (2,1),(2,3),(3,2),(3,3)} then R is ________________________Relation
4. If the principle diagonal elements in the relation matrix are all 1's ,then the matrix relation is___________________
5. Which of the following set is not a poset [ ]
   a) (R, ≤) b) (R, ≥) c) (R,=) d) (R, ≠)
6. Let R and S be any two equivalence relations on a non-empty set A. Which one of the following statement is true [ ]
   a) R∩S , R∪S are both equivalence relations
   b) R∪S is an equivalence relation
   c) R∩S is an equivalence relation
   d) neither R∩S nor R∪S is an equivalence relation
7. Consider the binary relation R ={ (x,y),(x , z),(z , x), (z , y)} on the set {x ,y , z} , which one of the following is true? [ ]
   a) R is symmetric but not anti-symmetric
   b) R is not symmetric but anti symmetric
   c) R is both symmetric and anti-symmetric
   d) R is neither symmetric nor anti symmetric
8. Which of the following is true. [ ]
   P: All totally ordered sets have least elements.
   Q: Hasse diagram of a totally ordered set is aline.
   a) P alone b) Q alone c) both P,Q d) neither P nor Q.
9. If R = {(x,y)/x>y} is a relation defined on A = {1,2,3,4} then the matrix of R is _________________
10. f : Z→Z defined by f(x) = x^3 then f is [ ]
    a) f is one-one b) f is into c) f is one-one and onto d) none of these
11. If A = { 3,4,5,6} and B ={ a,b} then the number of relations defined from A to B is
    a) 2^6 b) 2^8 c) 12 d) 8
12. Let f: B→C and g: A→B be two functions and let h = fog. Given that h is an onto function, which one of the following is True? [ ]
    a) f and g should both be onto functions.

b) f should be onto but g need not be onto
c) g should be onto but f need not be
d) both f and g need not be true.

13. The function f : Z → Z defined by $f(x) = x^2$ is _____ [ ]
a) one-one b) not one-one c) onto d) bijective

14. Which of the following function is not onto? [ ]
a) f (a , b) = a+ b b) f (a , b) = a c) f (a , b) = $|b|$ d) f (a , b) = a - b

15. Inverse of the function $f(x) = x^3 + 2$ is [ ]
a) $f^{-1}(y) = (y -2)^{1/2}$ b) $f^{-1}(y) = (y -2)^{1/3}$ c) $f^{-1}(y) = (y)^{1/2}$ d) $f^{-1}(y) = (y -2)$

SECTION-B

SUBJECTIVE QUESTIONS

1. Define partial order relation. Draw the Hasse diagram for the divisibility relation on the set A = {2, 3, 6, 12, 24, 36}.
2. Let X = {1, 2, 3, 4, 5, 6, 7} and R= {(x,y)/ x-y is divisible by 3} in X. show that R is an equivalence relation?
3. Let A be a given finite set and r(A) its power set. Let Í be the inclusion relation on the elements of r(A). Draw Hasse diagrams of <r(A), Í> for A={a}; A={a,b}; A={a,b,c} andA={a, b, c, d}.
4. Let f: R → R and g : R → R , where R is the set of real numbers . Find fog and gof , where $f(x) = x^2 -2$ and g(x) = x + 4 .State whether these functions are injective , surjective and bijective .
5. Let f: R → R be given by $f(x) = x^3 – 2$, Find f^{-1}?
6. Let f:Z→Z be a function defiled as $f(x) = x^2-3$. Is f a Bijective function? If not why?
7. Explain about initial functions and S.T f(x, y) = x * y is primitive recursive.
8. Let X={1,2,3} and f, g, h and s be functions from X to X given by f={<1,2>, <2,3>, <3,1>} , g={<1,2>, <2,1>, <3,3>}, h={<1,1>, <2,2>, <3,1>} and s={<1,1>, <2,2>, <3,3>}. Find fog, fohog, gos, fos.
9. Show that if eight people are in a room, atleast two of them have birthdays that occur on the same day of the week?
10. Apply is pigeon hole principle show that of any 14 integers are selected from the set S = {1, 2, 3…25} there are at least two where sum is 26. Also write a statement that generalize this result.

UNIT – III

Algebraic Structures

Objectives:

- To define various types of groups and study their properties
- To identify lattice and find their maximal and minimal elements.

Syllabus:

Algebraic Systems and Examples, general properties, semi group, Monoid, Groups, Subgroups, cyclic groups.

Sub Outcomes:

- Classify various types of algebraic structures
- Identify lattice for the given Poset
- Verify whether the given Lattice is distributive

Learning material

Algebraic structures:

Any system consisting of a set and n-array operations (+, - , *, o, etc.....) on the given set which is given algebraic structure.

Example: < S, *> be algebraic structure such that if a, b Є S, then a* b Є S.

General properties of Algebraic structure: Let **<** S, +>, < S, *> be algebraic structures then the properties are:

Closure property	if a , b Є S , then a + b Є S if a , b Є S, then a*b Є S
Associative property	if a , b , c Є S , then a + (b + c) = (a + b) + c Є S if a , b , c Є S , then a * (b * c) = (a * b) * c Є S
Identity property	if a Є S, then a + e = e + a = a, where 'e 'is the additive identity if a Є S, then a * e = e * a = a, where 'e 'is the multiplicative identity
Inverse property	if a , b Є S, then a + b = b + a = e , then 'b' is the additive inverse of a if a , b Є S, then a * b = b * a = e , then 'b' is the multiplicative inverse of a
Commutative property	if a , b Є S, then a + b = b + a if a , b Є S, then a * b = b * a

Distributive property:
Let < S, +, * > be algebraic structure, if a, b, c Є S, then a * (b + c) = (a * b) + (a * c) Є S.

Example:

- The set of natural numbers under addition i.e.< N, + > satisfies the closure , associative and commutative properties .
- The set consisting of 2x2 matrices with addition satisfies all above properties of algebraic structures.

Quasi group: A non-empty set G, with a binary operation '*' defined on it, such that it satisfies closure, is called 'Quasi Group'
Monoid:
A non-empty set G, with a binary operation '*' defined on it, such that it satisfies closure, associative, Identity is called 'Monoid '.
Semi Group:
A non-empty set G, with a binary operation '*' defined on it, such that it satisfies closure, associative, is called 'Semi Group'

Group:
A non-empty set G, with a binary operation '*' defined on it, such that it satisfies
closure, associative, Identity and Inverse is called 'Group' .

Abelian group:
A group G, with a binary operation '*' defined on it, such that it satisfies commutative property, is called 'Abelian group'.

Example: Show that the set G = { 1 , -1 , i ,-i } is group under multiplication ?

Solution: Consider the multiplication table

*	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

The table satisfies the
- Closure :- Since the all elements in the table belongs to the given set
- Associative :- If apply multiplication any three numbers the results are equal and belongs to the set
- Inverse : - Clearly $1^{-1} = 1$, $(-1)^{-1} = -1$, $i^{-1} = -i$, $(-i)^{-1} = i$ are the inverse elements.
- Identity :- '1' is the identity element in the set properties,

∴ The given set is the Group.

Group of integers modulo n: Consider the set of remainders when any non-negative integer is divided by n, a fixed positive integer. ie., Z_n = { 0,1,2,.......,n-1}
For all a, b $\in Z_n$, let $a \oplus_n b$ denote the remainder when a+b is divided by n.
Where the operation '$a \oplus_n b$' is known as 'addition modulo n'.

Example: Prove that the set {0, 1, 2, 3} is a finite abelian group under the operation addition modulo 4.

Solution: consider the addition modulo 4 table .

$\oplus_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

From the table, it shows that the given set with respect to $\oplus_4$ and elements of the set obeys

- **Closure :** - since all the elements in the table belongs to the given set
- **Associative :** - if we apply addition modulo 4' for any three numbers , the results are same ane belongs to the set
- **Identity:** - '**0'** is the **additive identity**
- **Inverse:** - Inverse of the each element is 0,1,2,3 are 0,3,2,1 respectively ,moreover , it is abelian since $a \oplus_4 b = b \oplus_4 a$.

Sub Group:
Let 'H' be a non – empty set which is subset to given group G, is said to be sub-group if it satisfies all the properties of the group .

Example: Prove that non- empty set H = {0, 2, 4} forms a sub-group of (Z_6, +) under addition.
Solution: We know that Z_6 = { 0,1,2,3,4,5} and then the addition modulo 6 table is

$+_6$	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

From the table

- **Closure : -** since all the elements in the table which belongs to the set
- **Associative : -** Consider $0 \equiv_{+} (2 \equiv_{+} 4(\text{mod} 6))$

$= 0 \equiv_{+} 0(\text{mod} 6)$

$= 0$

Consider $[(0 \equiv_{+} 2)(\text{mod } 6) \equiv_{+} 4(\text{mod } 6)]$

$= 2 \equiv_{+} 4(\text{mod} 6)$

$= 0$

H satisfies Associative property

- **Identity:** - H has identity '0' (from the table)
- **Inverse: -** $0^{-1} = 0$, $2^{-1} = 4$, $4^{-1} = 2$ are inverse of the H.

∴ H satisfies all the properties of Group.

∴ H is the subgroup of Group G.

Cyclic group:

A group (G,*) is said to be cyclic, if there exists an element a G such that every element og G can be written in the form a^n for some integer n.

UNIT-II
Assignment-Cum-Tutorial Questions
SECTION-A

Objective Questions

1. How many binary operations are possible on a set with n-elements
   A) 2^n B) 2^{n^2} C) n^{n^2} D) 2^{2^n} [ ]
2. Which of the following is a monoid
   A) (N,+) B) (N, x) C) (Z-{1}, x) D) (N-{1}, x)
3. Which of the following algebraic structure does not form a group
   A) (Z,+) Integers B) (R,+) Real numbers [ ]
   C) (R^+, x) Positive real numbers D) (N, x) Natural numbers.
4. Which of the following is not necessarily a property of a group is
   A) Commutativity B) Associativity [ ]
   C) Existence of inverse for every element D) Existence of identity.
5. Let the binary operation $*$ be defined in R by $a * b = 6ab$ then identity e=
   A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$ [ ]
6. The binary operation $\oplus$ on a set of integers is defined as $x \oplus y = x^2 + y^2$. Which one of the following statements is TRUE [ ]
   A) Commutative but not Associative
   B) B) Both Commutative and Associative
   C) Associative but not Commutative
   D) Neither Commutative nor Associative
7. The set G={1, 2, 3, 4, 5} under multiplication modulo 6 is [ ]
   A) An algebraic structure B) A non abelian group
   C) An abelian group C) None
8. The set {1, 2, 4, 7, 8, 11, 13, 14} is a group under multiplication modulo 15. The inverse of 4 and 7 are respectively [ ]
   A) 3 and 13 B) 2 and 11 C) 4 and 13 D) 8 and 14
9. The set {1, 2, 3, 5, 7, 8, 9} under multiplication modulo 10 is not a group. Given which are for possible reasons. Which one of them is false?
   A) It is not closed B) 2 does not has inverse[ ]
   C) 3 does not has inverse D) 8 does not has inverse

10. The inclusion of which of the following sets into S={ {1, 2},{1, 2, 3},{1, 3, 5},{1, 2, 4},{1, 2, 3, 4, 5} } is necessary and sufficient to make S a complete lattice under the partial order defined by set containment? [ ]

A) {1} B) {1}, {2,3}

C) {1}, {2,3} D) {1}, {1,3}, {1,2,3,4}, {1,2,3,5}

11. Which of the following is a semi group [ ]

A) $(N,*)$ with $a*b=a$ B) $(Z,\oplus)$ with $a\oplus b=a^3b^2$

C) $(Z,*)$ with $a*b=2a-b$ D) $(Q^+,*)$ with $a*b=\frac{a}{b}$

12. Let P={ {a},{b},{d},{a, b},{a, d},{c, d},{a, c, d},{b, c, d} } be the Poset under set inclusion as order. The greatest lower bound of { {a, c, d},{b, c, d} } is

A) {d} B) {c, d} C) {a} D) {b} [ ]

13. If G is a group of integers under addition and H is the subset consisting of all multiples of 3 then [ ]

A) H is a subgroup of G

B) H is not a subgroup of G as associative property does not hold

C) H is not a subgroup of G as H does not contain the identity element

D) None

14. Which of the following Binary operation is associative [ ]

A) In $(N,\star)$, $a\star b=a^2b$ B) In $(Z,\star)$, $a\star b=a^b$

C) In $(N,\star)$, $a\star b=a$ D) In $(N,\star)$, $a\star b=a-b$

SECTION-B

Descriptive Questions

1. Verify that R-{-1} of real numbers other than -1 is an abelian group with respect to the operation * defined by $a*b=a+b+ab$.
2. Show that the fourth root of unity forms a group and find out inverse of each element?
3. Show that the set of all positive rational numbers forms an abelian group under the composition defined by $a*b=\frac{ab}{4}$.
4. Let G= {-1, 0, 1}. Verify that G forms an abelian group under addition?
5. Prove that H = {0, 2, 4} forms a sub group of < Z6, +6>?
6. Show that the set G = { x/x = 2^a3^b and a, b ∈ Z} is a group under multiplication
7. Define Lattice. Verify that the poset {(1, 5, 25, 125), /} is a lattice or not.
8. A binary operation * is defined on set of integers Z by $a*b=a+b-ab$, for all *a* and *b* in Z. Show that (Z, *) is a semi group.

9. Show that the fourth roots of unity forms a group under usual multiplication and find out inverse of each element.
10. Consider the group G = {1, 2, 4, 7, 8, 11, 13, 14} under multiplication modulo 15.Construct the multiplication table of G?
11. If G is a group such that $(ab)^m = a^m b^m$ for three consecutive integers m for all *a, b* ∈ G, show that G is abelian.
12. The set of integers Z, is an abelian group under the composition defined by ⊕ such that a⊕ b = a + b+ 1 for a, b ∈ Z. Find
    i) the identity of (Z, ⊕) and
    ii) Inverse of each element of Z.
13. Consider the group, G = {1, 2, 4, 7, 8, 11, 13, 14} under multiplication modulo 15:
    (a) Construct the multiplication table of G.
    (b) Find the values of: 2^{-1}, 7^{-1} and 11^{-1}.
    (c) Find the orders and subgroups generated by 2, 7, and 11.
14. The set 'S' of all ordered pairs (a, b) of real numbers for which a 6= 0 w.r.t. the operation × defined by (a, b) × (c, d) = (ac, bc+d) is a group. Find (i) the identity of (G, o) and (ii) Inverse of each element of G.

UNIT – IV

Graph Theory - I

Objectives:

- Classify the concepts and properties of graphs
- Illustrate the concept of isomorphism.

Syllabus: Concepts of graphs, subgraphs, multi-graphs, matrix representation of graphs, adjacency matrices, incidence matrices, isomorphic graphs.

Outcomes:

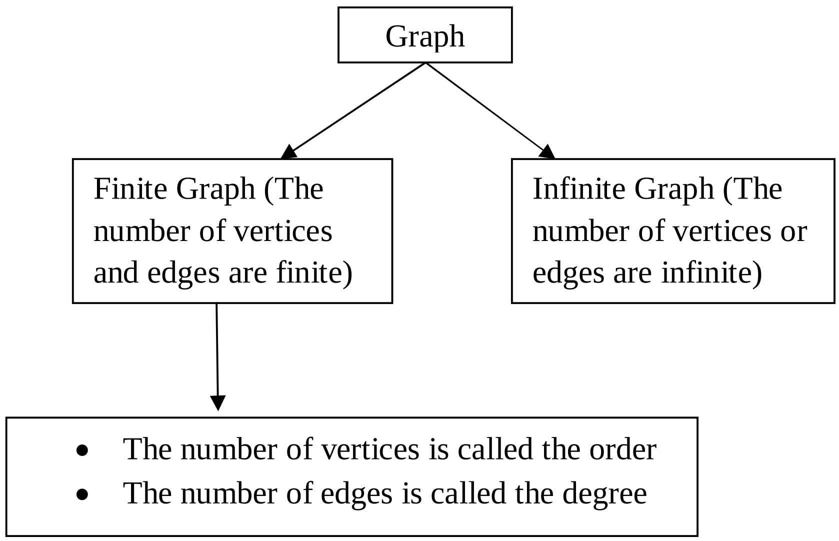
Student will be able to

- Identify the adjacency and incidence matrices for the given graphs.
- testwhether the given graphs are isomorphic or not

Learning Material

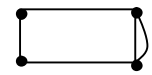
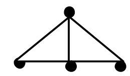
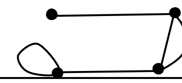
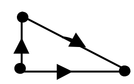
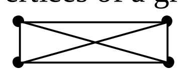
Graph:

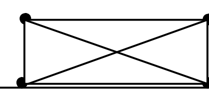
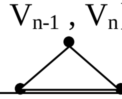
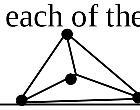
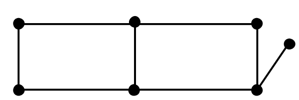
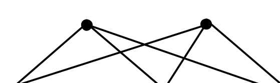
A graph G consists of a set V of vertices and a collection of edges (unordered pair of vertices) and is symbolically represented as G (V, E).



- An edge of a graph that joins a vertex to itself is called Loop.
- Two or more edges that join the same pair of distinct vertices are called multiple edges.
- Any two vertices connected by an edge are called adjacent vertices otherwise they are called isolated vertices.
- The edge 'e' that joins the vertices u and v is said to be incident on each of its end points u and v.
- The sum of the degrees of vertices of a graph G is equal to the twice the number of edges (Handshaking Theorem).

Finite Graph

- **Trivial Graph** – A graph with only one vertex and no edges.
- **Multigraph** – A graph with no loops

- **Simple Graph** –A graph which has neither loops nor multiple edges

- **Pseudograph** - A graph which has loops and multiple edges

- **Directed Graph** – A graph in which each edge has a direction

- **Regular Graph** – If all vertices of a graph have the same degree

- **Null Graph** - It contains only an isolated node (The edge set is empty)

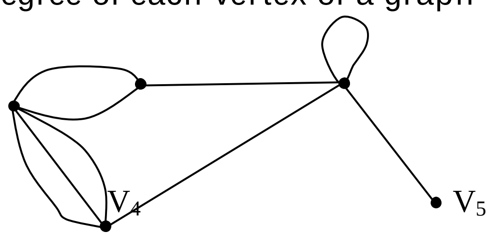
- **Complete Graph** – Every vertex in a Graph G is connected with every other vertex

- **Cycle** – It consists of n vertices $V_1, V_2, V_3, \ldots, V_n$ and edges $\{V_1, V_2\}, \{V_2, V_3\}, \ldots, \{V_{n-1}, V_n\}$ and $\{V_n, V_1\}$.

- **Wheel** – when an additional vertex is added to the cycle and this new vertex is connected to each of the n vertices in cycle by the new edges

- **Bipartite Graph** – If the vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge in E connects a vertex in V_1 and vertex in V_2

- **Complete Bipartite Graph** – A graph whose vertex set is partitioned into subsets V_1 and V_2 in which there is an edge between each pair of vertices.


Degree of a vertex:

- The degree of a vertex of an **undirected graph** is equal to the number of edges in G which contains the vertex and is denoted by deg (v)
  - A vertex of degree ‘0’ is called an isolated vertex.
  - A degree of degree ‘0’ is called an end vertex (A vertex is pendent iff it has a degree ‘1’).

Example: Find the degree of each vertex of a graph

V_1 V_2 V_3



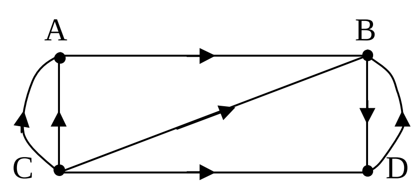
deg (V_1) = 5 ; deg (V_2) = 3 ; deg (V_3) = 5 ; deg (V_4) = 4 ; deg (V_5) = 1

- The degree of a **directed graph** is given by

Total deg (V) = Indeg (V) + outdeg (V)

  - The number of edges ending at V is called the in-degree of the vertex of a directed graph and is denoted by Indeg (V) or $deg^-(V)$.
  - The number of edges beginning at V is called the out-degree of the vertex of a directed graph and is denoted by outdeg (V) or $deg^+ (V)$.
  - A vertex with zero indegree is called source.
  - A vertex with zero outdegree is called sink.

Example: Find the degree of each vertex of a digraph



Indeg (A) = 2 outdeg (A) = 1 Totaldeg (A) = 3

Indeg (B) = 3 outdeg (B) = 1 Totaldeg (B) = 4

Indeg (C) = 0 outdeg (C) = 3 Totaldeg (C) = 3

Indeg (D) =2 outdeg (D) = 1 Totaldeg (D) = 3

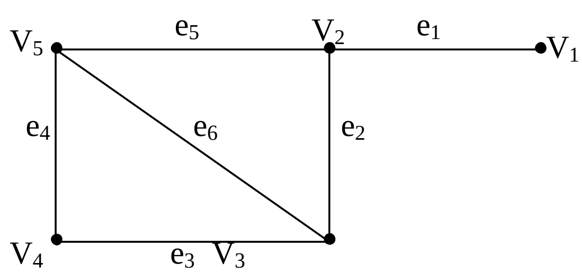
....

Representation of Graphs:
- Matrix Representation
- Linked Representation

Matrix Representation
- Undirected Graph
  - **Adjacency Matrix:**
    $A = [a_{ij}] = 1, (V_i, V_j)$ □ G
    = 0, otherwise
  - **Incidence Matrix:**
    $I = [a_{ij}] = 1$, e_j is incident on V_i
    = 2, e_j is a loop at V_i
    = 0, otherwise
- Directed Graph
  - **Adjacency Matrix:**
    $A = [a_{ij}] =1, (V_i, V_j)$ □ D
    = 0, otherwise
  - **Incidence Matrix:**
    $I = [a_{ij}] = 1$, e_j is incident out of V_i
    = -1, e_j is incident into V_i
    = 0, e_j is not incident on V_i

Examples:

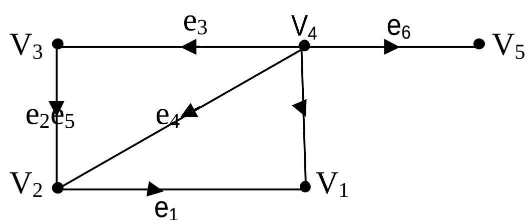
➢ Find the Adjacency and Incidence Matrices for the following graph



Sol: The adjacency and incidence matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

- Find the Adjacency and Incidence Matrices for the following digraph



Sol: The adjacency and incidence matrices are

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{and} \quad I = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

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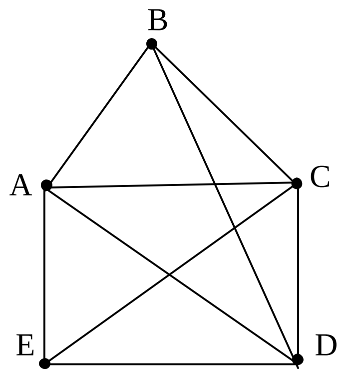
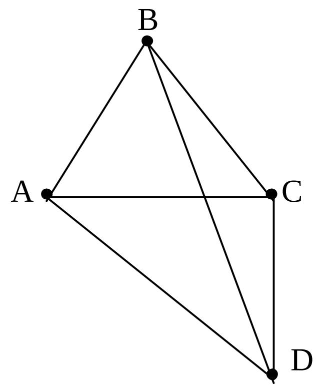
Subgraph – It is obtained by removing certain vertices and edges from the given graph

- The subgraph of G obtained by deleting V (vertex) and all the edges incident on V is called the **Vertex Deleted Subgraph** of G
- The subgraph of G obtained by deleting e (edge) is called the **Edge Deleted Subgraph** of G
- Suppose G(V, E) is a graph and $G_1(V_1, E_1)$ is a subgraph of G such that $V_1 = V$ then G_1 is called a **Spanning Subgraph**
- Suppose G(V, E) is a graph and $G_1(V_1, E_1)$ is a subgraph of G such that every edge of G is an edge of G_1then G_1 is called a **Subgraph of G Induced by V_1**

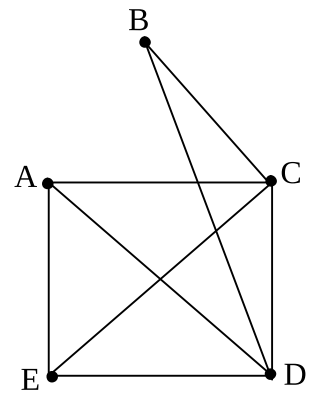
Example:



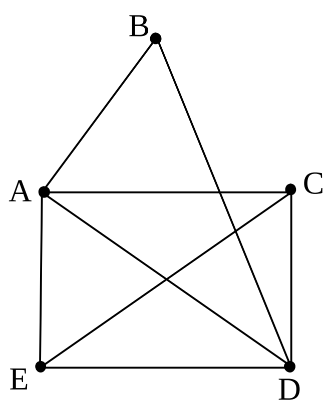
Graph (G)



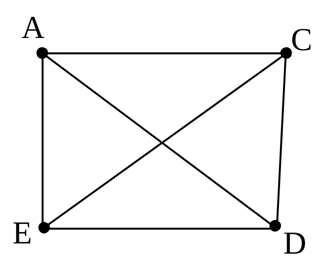
Subgraph (G-V)



Subgraph (G-e)



Spanning Subgraph



Induced Subgraph

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Isomorphism:

Two graphs G and G^1 are said to be isomorphic if there is a one to one correspondence between their vertices and edges such that adjacency of vertices is preserved and is denoted by $G \cong G^1$.

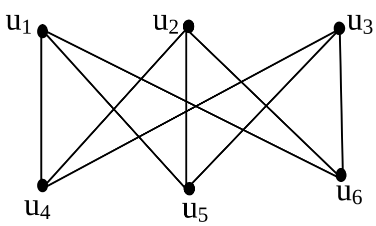
- Adjacency preserved – For any vertices u, v in G, if u and v are adjacent in G the the corresponding vertices u^1 and v^1 are also adjacent in G^1

Working rule to verify the Isomorphism of graphs:

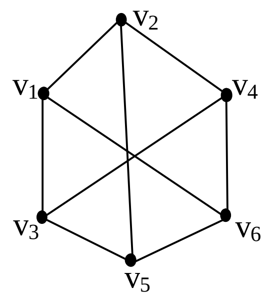
- Verify that the Graphs G and G^1 have equal number of vertices and equal number of edges or not
- If they are equal then calculate the degree of each vertex in both the graphs
- Finally verify the adjacency depending on the degree of vertices

Problem:

- Verify that the following graphs are isomorphic or not



G



G^1

Solution:

G	G^1
No.of vertices = 6	No.of vertices = 6
No. of edges = 9	No. of edges = 9
deg (u_1) = 3	deg (v_1) = 3
deg (u_2) = 3	deg (v_2) = 3
deg (u_3) = 3	deg (v_3) = 3
deg (u_4) = 3	deg (v_4) = 3
deg (u_5) = 3	deg (v_5) = 3
deg (u_6) = 3	deg (v_6) = 3

Thus $u_1 = v_1$, $u_2 = v_2$, $u_3 = v_3$, $u_4 = v_4$, $u_5 = v_5$, $u_6 = v_6$ i.e., the adjacency is preserved.

Hence G and G^1 are isomorphic

UNIT-IV
Assignment-Cum-Tutorial Questions
SECTION-A

Objective Questions

1. In a simple graph with p+1 vertices, the maximum degree of any vertex is
   a) p+1 b) p c) p-1 d) p-2 [ ]
2. Which of the following degree sequences cannot represent an undirected graph?
   i.{3,4,2,2} ii. {3,1,2,2} iii.{1,4,2,2,3,5} iv.{5,5,4,.4}
   a) iv only b) i and iii c) iii only d) ii and iv [ ]
3. If a graphG contains 21 edges, 3 vertices of degree 4 and the other vertices of degree 3 then the number of vertices of G are _________
4. Define Regular , connected graphs?
5. A vertex of degree zero is called______________________
6. In any graph the number of vertices of odd degree is ___________________
7. Draw the cycle graph of order 5?
8. Draw the wheel graph of order 4?
9. Draw the graph which is both cycle and bipartite graph?
10. The minimum number of edges in a connected graph having 19 vertices is
    a)19 b)20 c)17 d)18 [ ]
11. Which of the following statements is/are true for undirected graph
    P: Number of odd degree vertices is even
    Q: Sum of degrees of all vertices is even [ ]
    a) P only b)Q only c) Both P and Q d) Neither P and Q
12. A pendent vertex has degree equal to [ ]
    a) 0 b) 1 c) 2 d) 3

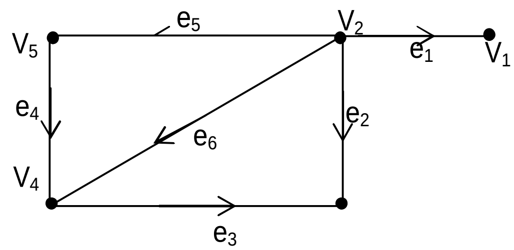
SECTION-B

Descriptive Questions

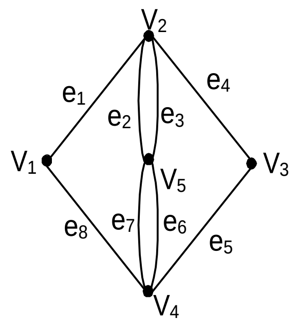
1. Is the following sequence is degree sequence? If so, find the graph? 1,1,2,2,2,3,3,4?
2. Draw the graphs of $K_{2,5}$ and $K_{3,3}$.
3. Consider the digraph G = (V, E) where V = {a, b, c, d, e} and E = {(a,c), (b,a), (b,b),(b,c),(c,d),(c,e),(d,c),(d,d),(e,b) }. Draw the graph G and also find the degrees of vertices inG.
4. Definegraph. LetGbeanon - directedgraphoforder9suchthateachvertexhasdegree5or 6. Prove that at

least 5 vertices have degree 6 or at least 6 vertices have degree 5.

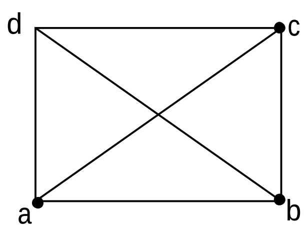
5. Find all indegree and outdegree of the nodes of the following graph

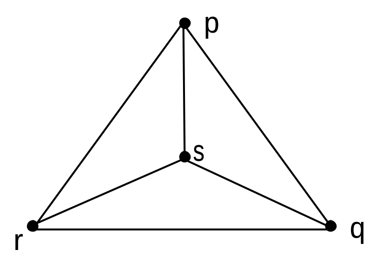


6. Find the adjacency and incidence matrices for the following graph.

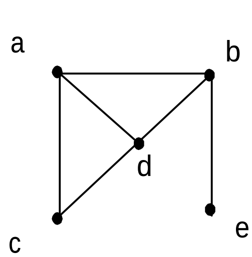


7. Compare whether the following graphs are Isomorphic or not?

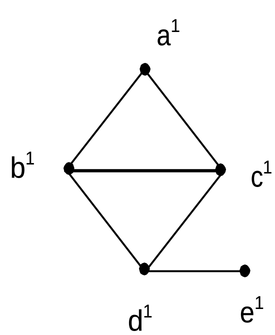




8. When we say that the graphs G1 and G2 are isomorphic and verify whether the following graphs are isomorphic ornot.

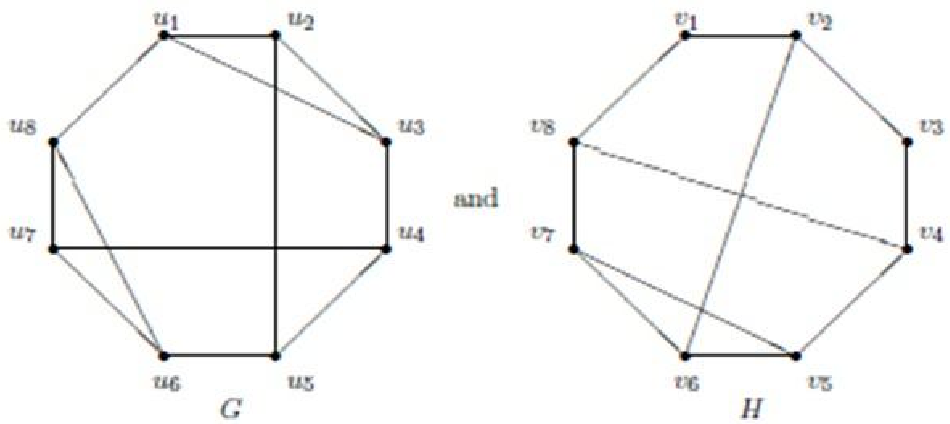


G_1



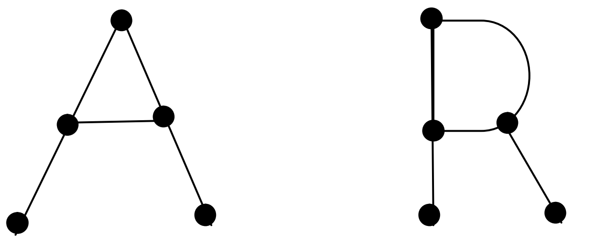
G_2

9. Determine the following graphs isomorphic or not? Justify your answer.

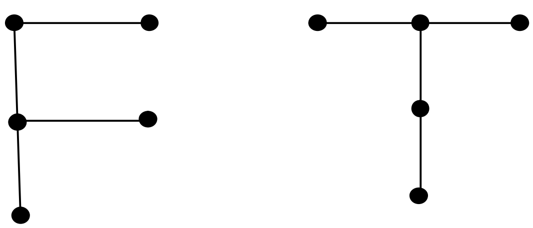


10. Which among the following pairs are Isomorphic

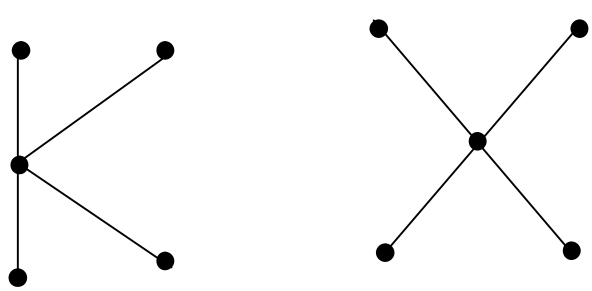
I.



II.



III.



Section C.

Gate Questions:

1. Suppose theadjacency relation of vertices in a graph is represented in a table as adj(X,Y). Which of the following queries cannot be expressed by a relational algebra expression of constant length? [ ]

(a) List all vertices adjacent to a given vertex
(b) List allvertices which have self loops
(c) List all vertices which belong to cycles of less than three vertices
(d) List all vertices reachable from a given vertex. [GATE 2001]

2. How many undirected graphs (not necessarily connected) can be constructed out of a given set $V = \{v_1, v_2, \ldots, v_n\}$ of n vertices? [ ]

(a) $n(n-1)/2$ (b) 2^n (c) n! (d) $2^{n(n-1)/2}$ [GATE 2001]

3. Maximum number of edges in a n – node undirected graph without self loops is [ ]

(a) n^2 (b) $n(n-1)/2$ (c) n-1 (d) $n(n+1)/2$ [GATE 2002]

4. Let G be a directed graph whose vertex set is the set of numbers from 1 to 100. There is an edge from a vertex i to vertex j iff either j = i+1 or j = 3i. The minimum number of edges in a path in G from vertex 1 to vertex 100 is [ ]

(a) 4 (b) 7 (c) 23 (d) 99 [GATE 2005]

5. Which of the following statements is/are TRUE for undirected graphs? [ ]

P: Number of odd degree vertices is even.
Q: Sum of degrees of all vertices is even.

(A) P only (B) Q only (C) Both P and Q (D) Neither P nor Q [GATE 2013]

6. An ordered n-tuple $(d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \ldots \geq d_n$ is called graphic if there exists a simple undirected graph with n vertices having degrees $d_1, d_2, \ldots, d_n$ respectively. Which of the following 6-tuples is NOT graphic? [ ]

(A) (1, 1, 1, 1, 1, 1) (B) (2, 2, 2, 2, 2, 2)
(C) (3, 3, 3, 1, 0, 0) (D) (3, 2, 1, 1, 1, 0) [GATE 2014]

7. The maximum number of edges in a bipartite graph on 12 vertices is ________________. [GATE 2014]

UNIT – V
GRAPH THEROY -2

Objectives:

- To find Hamiltonian graph and Euler graph from the given graph
- Identify the planar graphs from the given graph
- Graph coloring

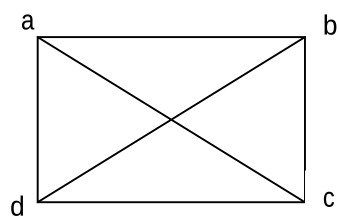
Syllabus: Hamiltonian Graph, Planar Graphs, Chromatic number of a graph.

Sub Outcomes:

- use the concepts of graph theory to provide solutions for routing applications in computer networks
- identify the Hamiltonian graph
- identify the Euler graph

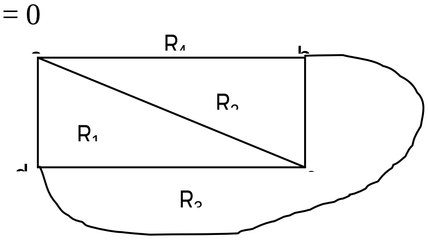
Learning Material

Hamiltonian path: It is a path in a graph which covers all vertices without repetition
Hamiltonian cycle: It is a closed Hamiltonian path.
Hamiltonian graph: A graph is said to have Hamiltonian graph if it is either Hamiltonian path or Hamiltonian cycle.
Ore's theorem: A graph of n vertices is said to have Hamiltonian cycle if each vertex has degree n/2 or more.
Grinberg theorem: Let G be a simple graph with no crossing of edges and if $\sum(i-2)(r_i - r_i^1) = 0$.Then G has Hamiltonian cycle. Where i denote no of edges of the region r_i.
Problem 1: Find whether the following graph has Hamiltonian cycle?



Solution: In the given graph, there are no crossing edges and then by Grinbergs's theorem

we have 4 regions bounded by the 3 edges and 1 region bounded by the 4 edges.
$(3-2)(r_3 - r_3^1)+(4-2)(r_4 - r_4^1) = 0$



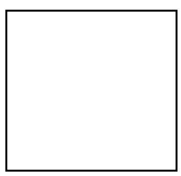
The number of possibilities are for 4 regions are

0 + 4 = 4 X
1 + 3 = 4 X
2 + 2 = 4 √
3 + 1 = 4 X
4 +0 = 4 X

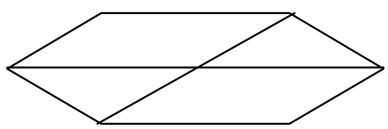
∴ The graph G has Hamiltonian cycle that is abcda.

Planar graph: A graph G is said to be planar if it can be drawn in a plane such that no two edges cross each other.

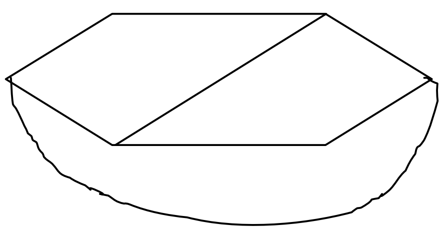
Example:



it is a planar graph



It is a planar graph because there are no crossing edges.

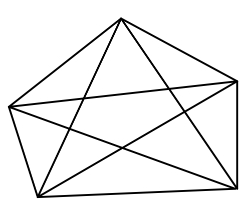


Euler theorem: If a connected graph G is planar if $|v| - |E| + |R| = 2$.
where $|v|$ is the no. of vertices, $|E|$ is the no. of edges and $|R|$ no. of regions.

Notes:

- In a connected planar graph, $|E| \leq 3|v| - 6$.
- In a connected planar graph, $|R| \leq 2|v| - 4$.

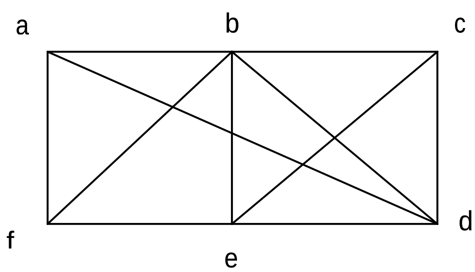
Problem 2: Determine whether the graph k_5 is planar?
Solution:



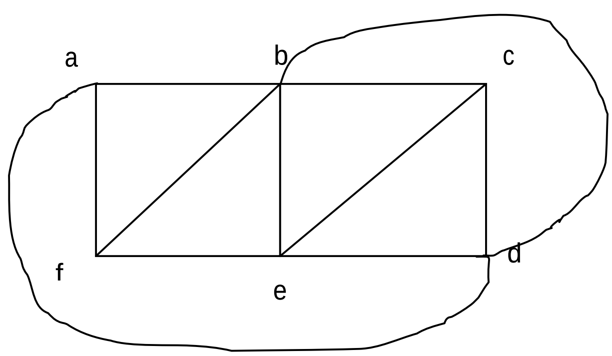
In the obtained graph, there are 5 vertices, 10 edges and by the properties of planar graph, it didn't satisfy the properties of planar graph.
$\therefore$ k_5 is not a planar graph.

Kuratowski's theorem:-
A simple graph is planar if and if only it does not contain k_5 or $k_{3,3}$ as sub graphs.

Problem 3: Is the following graph is planar?



Solution: In this graph, there are 6 vertices and 11 edges and it satisfies the properties of the planar graph, then the given graph is planar graph.
The given graph can be written as



Coloring

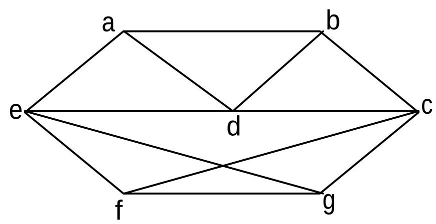
Vertex colouring is the assignment of colours to the vertices such that no two adjacent vertices have same colour

Edge colouring is the assignment of colours to the edges such that no two adjacent edges have same colour.

Chromatic number: - The minimum number of colours needed to vertex colouring is called the 'Chromatic number' and is denoted by $\chi(G)$.

- $\chi(G) \leq |v|$
- $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the largest degree of vertex of G.
- $\chi(G) \geq |v|/(|v| - \delta(G))$, where $\delta(G)$ is the smallest degree vertex of G.

Problem 4: Find the chromatic number for the following graph?

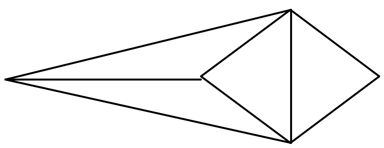


Solution: In this graph, $|v| = 7$, $\therefore \chi(G) \leq 7$ and $\Delta(G) = 4$ and $\delta(G) = 3$
then by properties, $\therefore \chi(G) \leq 7$ and $\chi(G) \geq 7/4$. Thus $\chi(G)$ is either 2 or 3 or 4 or 5.
$\therefore \chi(G) = 4$.

Chromatic Index:- The minimum number of colours needed to edge colouring is called 'Chromatic index' and is denoted by $\chi^1(G)$.

Vizing theorem: If G is a simple graph with maximum vertex degree $\Delta(G)$, then
$\Delta(G) \leq \chi^1(G) \leq \Delta(G) + 1$.

Problem 5: Find the chromatic index for the following graph



Solution: From Vizing theorem, $\Delta(G) \leq \chi^1(G) \leq \Delta(G) + 1$.
and $\Delta(G) = 4$. Chromatic index is 4.

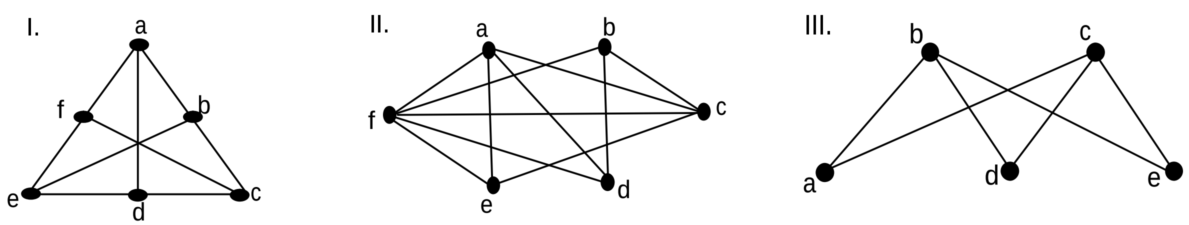
UNIT-IV
Assignment-Cum-Tutorial Questions
SECTION-A

Objective Questions

1. Euler formula for planar graphs is________________________________
2. Chromatic number for wheel graph w_n is _________________________
3. Give an example of a graph which is Hamiltonian but not Eulerian graph?
4. The Hamiltonian cycle for the complete bipartite $K_{2,3}$ is ___________________
5. The chromatic number of a graph $k_{m,n}$ is____________________
6. The chromatic number of a wheel graph of six vertices is _____________________

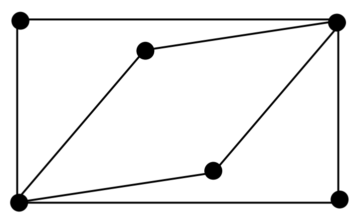
7. Suppose G is a connected planar graph with 12 regions of degree 3 then the no. of vertices=______________ [ ]
   a) 4 b)8 c)12 d)10

8. Which of the following can be represented as plane graphs [ ]



a)I only b) I and II only c) II and III only d) None

9. Which among the following is true about the graph given below [ ]

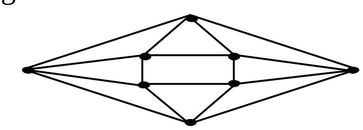


a)Eulerian and Non Hamiltonian b) Hamilton and Non Eulerian
c) Non Eulerian and Non Hamiltonian d) None

10. Let G be the non planar graph with minimum possible number of edges. Then G has
    a) 9 edges and 5 vertices b) 9 edges and 6 vertices [ ]

    b) 10 edges and 5 vertices d) 10 edges and 6 vertices

11. The minimum number of colors required to color the following group such that no two adjacent vertices are assigned the same color is [ ]



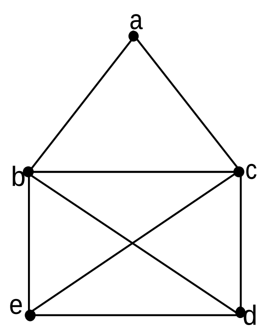
a) 2 b) 3 c) 4 d) 5

12. The chromatic number of a complete graph of five vertices is _________ [ ]
 a)3 b)4 c)5 d)7
13. A tree with 12 vertices has _____________edges [ ]
 a) 10 b)11 c)12 d)13
14. Which of the following is true
 I. Every tree with at least one edge must has at least two pendent vertices
 II. Every tree is a planar graph
 III. Every tree is bipartited
 a) I only b) II and III only c) I, II and III only d) I and II only

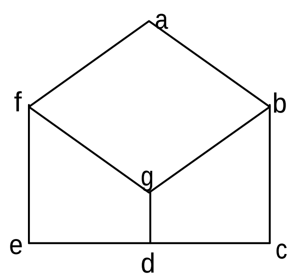
SECTION-B

Descriptive Questions

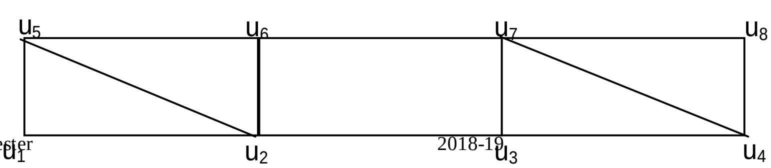
1. Prove that the following graph has Hamiltonian cycle.



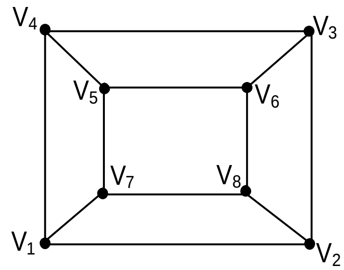
2. Find whether the following graph has Hamiltonian cycle or not? Is the graph hamiltonian graph?



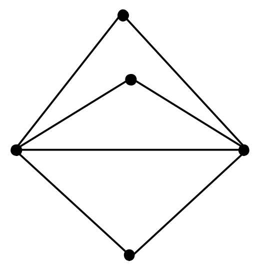
3. Find whether the following graph has Hamiltonian cycle?



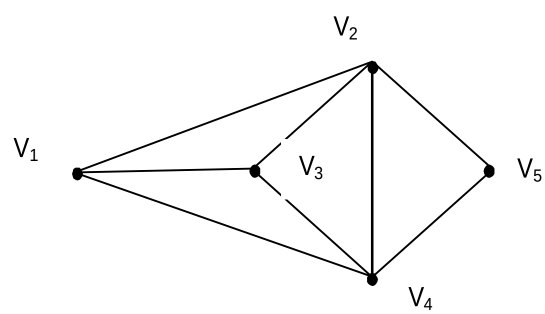
4. Find the Hamilton circuit for the following graph?



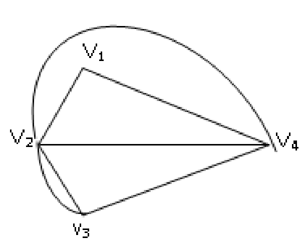
5. Find the chromatic number of the following graph



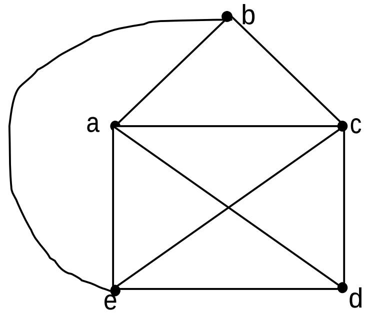
6. Define chromatic number. Find the chromatic number for the following graph.



7. Draw the bipartite graph $K_{3,3}$ and find its chromatic number.
8. Prove whether K_4 and K_5 are planar or non-planar.
9. Find the Euler path to the following graph.



10. Chek the following graph is Eulerian graph or not? If so find Eulerian trail or Eulerian circuit.



11. Draw a graph with six vertices which is Eulerian graph.

Section C.

Gate Questions:

1. Common Data Question:

The 2^n vertices of a graph G corresponds to all subsets of a set of size n, for n >= 6 . Two vertices of G are adjacent if and only if the corresponding sets intersect in exactly two elements.

1. The number of vertices of degree zero in G is: [ ]

(a)1 (b) n (c) n+1 (d)2^n

2. The maximum degree of a vertex in G is: [ ]

(a)$(n/2)C2x2^{n/2}$ (b) 2^{n-2} (c) $2^{n-3}x3$ (d) 2^{n-1}

3. The number of connected components in G is: [ ]

(a)n (b) n+2 (c) $2^{n/2}$ (d) $2^n/n$ [GATE 2006]

2. Let G be the non-planar graph with the minimum possible number of edges. Then G has

(a)9 edges and 5 vertices (b) 9 edges and 6 vertices [ ]

(c) 10 edges and 5 vertices (d) 10 edges and 6 vertices [GATE 2007]

3. The height of a binary tree is the maximum number of edges in any root to leaf path. The maximum number of nodes in a binary tree of height h is: [ ]

(a)2^h - 1 (b) 2^{h-1} -1 (c) 2^{h+1} -1 (d) 2^{h+1} [GATE 2007]

4. The maximum number of binary trees that can be formed with three unlabeled nodes is:

(a)1 (b) 5 (c) 4 (d) 3 [GATE 2007] [ ]

5. What is the largest integer m such that every simple connected graph with n vertices and n edges contains at least m different spanning trees? [ ]

(a) 1 (b) 2 (c) 3 (d) n [GATE 2007]

6. G is a simple undirected graph. Some vertices of G are of odd degree. Add a node v to G and make it adjacent to each odd degree vertex of G. The resultant graph is sure to be

(a) regular (b) complete (c) Hamiltonian (d) Euler [GATE 2008]

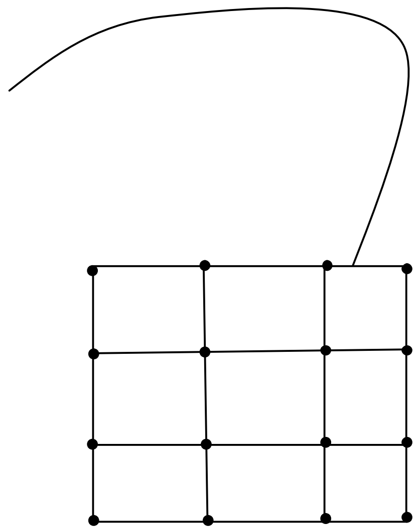
7. Which of the following statements is true for every planar graph on n vertices?

(a) The graph is connected (b) The graph is Eulerian [ ]

(c) The graph has a vertex-cover of size at most 3n/4

(d) The graph has an independent set of size at least n/3 [GATE 2008]

8. What is the chromatic number of the following graph [ ]

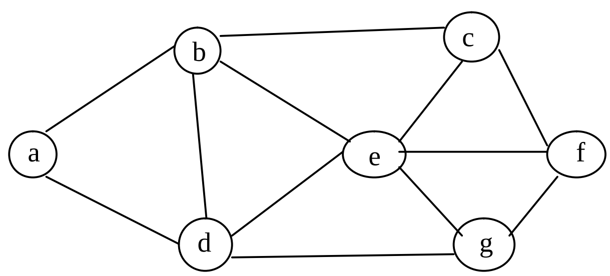


(a) 2 (b) 3 (c) 4 (d) 5 [GATE 2008]

9. Consider the following sequence of nodes for the undirected graph given below. [ ]

1. a b e f d g c
2. a b e f c g d
3. a d g e b c f
4. a d b c g e f

A Depth First Search (DFS) is started at node a. The nodes are listed in the order they are first visited. Which all of the above is (are) possible output(s)?



(a) 1 and 3 only (b) 2 and 3 only
(c) 2, 3 and 4 only (d) 1, 2 and 3 [GATE 2008]

10. What is the chromatic number of an n-vertex simple connected graph which does not contain any odd length cycle? Assume $n \geq 2$

(a) 2 (b) 3 (c) n – 1 (d) n [GATE 2009]

11. Which one of the following is TRUE for any simple connected undirected graph with more than 2 vertices? [ ]

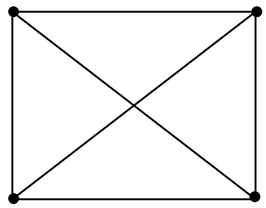
(a) No two vertices have the same degree
(b) At least two vertices have the same degree
(c) At least three vertices have the same degree
(d) All vertices have the same degree [GATE 2009]

12. In a binary tree with n nodes, every node has an odd number of descendants. Every node is considered to be its own descendant. What is the number of nodes in the tree that have exactly one child? [ ]

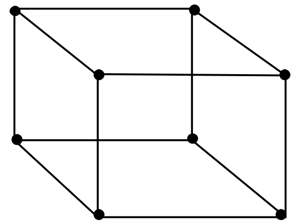
(a) 0 (b) 1 (c) (n -1) / 2 (d) n-1 [GATE 2010]

13.

K_4



Q_3



(a) K_4 is planar while Q_3 is not
(b) Both K_4 and Q_3 are planar
(c) Q_3 is planar while K_4 is not
(d) Neither K_4 nor Q_3 are planar [GATE 2011]

14. Let G be a simple undirected planar graph on 10 vertices with 15 edges. If G is a connected graph, then the number of bounded faces in any embedding of G on the plane is equal to
(a) 3 (b) 4 (c) 5 (d) 6 [GATE 2012]

15. Let G = (V,E) be a directed graph where V is the set of vertices and E the set of edges. Then which one of the following graphs has the same strongly connected components as G? [ ]
(a) $G_1 = (V, E_1)$ where $E_1 = \{(u,v) / (u,v) \notin E\}$
(b) $G_2 = (V, E_2)$ where $E_2 = \{(u,v) / (u,v) \notin E\}$
(c) $G_3 = (V, E_3)$ where $E_3 = \{(u,v)$ / there is a path of length ≤ 2 from u to v in E$\}$
(d) $G_4 = (V_4, E)$ where V_4 is the set of vertices in G which are not isolated
[GATE 2014]

16. If G is a forest with n vertices and k connected components, how many edges does G have? [ ]
(a) [n / k] (b) [n / k] (c) n -k (d) n – k+ 1 [GATE 2014]

17. A binary tree T has 20 leaves. The number of nodes in T having two children is
(a) 18 (b) 19 (c) 17 (d) any number between 10 and 20
[GATE 2015]

18. The height of a tree is the length of the longest root-to-leaf path in it. The maximum and minimum number of nodes in a binary tree of height 5 are [ ]
(a) 63 and 6, respectively (b) 64 and 5, respectively
(c) 32 and 6, respectively (d) 31 and 5, respectively [GATE 2015]

19. The minimum number of colours that is sufficient to vertex-colour any planar graph is _____________.
[GATE 2016]

20. Let T be a binary search tree with 15 nodes. The minimum and maximum possible heights of T are: [ ]

Note: The height of a tree with a single node is 0.
(a) 4 and 15, respectively (b) 3 and 14, respectively
(c) 4 and 14, respectively (d) 3 and 15, respectively [GATE 2017]

21. Let T be a tree with 10 vertices. The sum of the degrees of all the vertices in T is _____. [ ]
(a) 18 (b) 19 (c) 20 (d) 21 [GATE 2017]

22. G is undirected graph with n vertices and 25 edges such that each vertex has degree at least 3. Then the maximum possible value of n is ________ [ ]
(a) 4 (b) 8 (c) 16 (d) 24 [GATE 2017]

23. L et *G* be a simple undirected graph. Let T_D be a depth first search tree of *G*. Let T_B be a breadth first search tree of *G*. Consider the following statements. [ ]
(I) No edge of *G* is a cross edge with respect to T_D. (A cross edge in *G* is between two nodes neither of which is an ancestor of the other in T_D.)
(II) For every edge (u,v) of *G*, if u is at depth i and v is at depth j in T_B, then $|i - j| = 1$.
Which of the statements above must necessarily be true?
(A) I only (B) II only (C) Both I and II (D) Neither I nor II
[GATE 2018]

Unit – VI

Recurrence Relations

Learning Material

Objectives:

- Find the solution of linear recurrence relation with constant coefficients.

Syllabus:

Recurrence Relations- Formulation, Solving linear homogeneous recurrence Relations by substitution, The Method of Characteristic Roots, Solving Inhomogeneous Recurrence Relations

Outcomes:

- Use the concept of recurrence relations in certain counting problems.

Recurrence Relation(RR): An equation that expresses a_n in terms of one or more previous terms of the sequence namely $a_0, a_1, a_2, \ldots., a_{n-1}$ for all integers $n \geq 1$.

These recurrence relations are divided into two types.

1. Linear Recurrence Relation.
2. Non Linear Recurrence Relation.

Linear Recurrence Relation: A RR of the form $C_0(n)a_n + C_1(n)a_{n-1+} C_2(n)a_{n-2+\ldots+} C_k(n)a_{n-k} = f(n)$ for $n \geq k$ is said to be a Liner RR.

- If $C_0(n), C_1(n), \ldots, C_k(n)$ and $f(n)$ are functions of n, this relation is called linear RR with variable coefficients.
- The order of a RR is the difference between the largest and the smallest subscripts appearing in the relation.
- If $C_0(n), C_1(n), \ldots, C_k(n)$ and $f(n)$ are constants, this relation is called linear RR with constant coefficients.
- If $C_0(n), C_k(n)$ are not identically zero, k is called the degree of that linear RR.
- If $f(n) = 0$, this equation is called homogeneous linear RR otherwise non homogeneous RR.

Solution of a RR: A sequence $\{a_n\}_{n=0}^{\infty}$ is said to be a solution of a RR if each value of a_n i.e., $a_0, a_1, \ldots, a_n$ satisfies the RR.

In general these RR's are solved by using three methods.

1. Substitution or Iteration Method.
2. Method of characteristic Roots.
3. Generating Functions.

Substitution method: In this method, the RR for a_n is used repeatedly to solve for a general expression a_n in terms of n.

Problem: Solve the RR $a_n = a_{n-1} + 2$; $a_0 = 3$

Sol: $a_1 = a_0 + 2 = 3+2 = 3+(1x2)$

$a_2 = a_1 + 2 = (3+2)+2 = 3+(2x2)$

$a_3 = a_2 + 2 = (3+2x2)+2 = 3+(3x2)$

⋮

$a_n = a_{n-1}+2 = 3+ (n-1)2 = 3+2n$

Method of characteristic Roots:

The general solution of RR is $a_n = a_n^{(h)} + a_n^{(p)}$

Steps for $a_n^{(h)}$ (solution of homogeneous part):

- Write the characteristic equation of the given RR and then find its roots.
  - If the roots are real and distinct then $a_n = C_1\alpha_1^n + C_2\alpha_2^n + C_3\alpha_3^n + ...$
  - If the roots are real and equal ($\alpha_1=\alpha_2$)then $a_n = (C_1 + C_2 n)\alpha_1^n + C_3\alpha_3^n + ...$
  - If the roots are in complex form i.e., α_1= a+ib, α_2= a-ib then

    $a_n = r^n(C_1 \cos n\theta + C_2 \sin n\theta)$ where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Steps for $a_n^{(p)}$:

- Suppose f(n) is the polynomial of degree ‘q’ and ‘1’ is not a root of the characteristic equation then $a_n^{(p)} = A_0 + A_1 n + A_1 n^2 + ... + A_q n^q$
- Suppose f(n) is the polynomial of degree ‘q’ and ‘1’ is a root of multiplicity ‘m’ of the characteristic equation then $a_n^{(p)} = n^m(A_0 + A_1 n + A_1 n^2 + ... + A_q n^q)$
- Suppose f(n) = α b^n where α is a constant and b is not a root of the characteristic equation then $a_n^{(p)} = A_0 b^n$
- Suppose f(n) = α b^n where α is a constant and b is a root of multiplicity ‘m’of the characteristic equation then $a_n^{(p)} = A_0 b^n n^m$

Where A_0, A_1, A_2, …, A_q are constants and are to be evaluated by the fact that $a_n = a_n^{(p)}$ whch satisfies the given RR.

Problem: Solve the RR $a_{n+2}+3a_{n+1}+2a_n = 3^n$ for $n \geq 0$.

Sol: *To find* $a_n^{(h)}$ *:* The characteristic equation for the homogeneous part of the given relation is

$k^2 + 3k + 2 = 0$ and the roots are -1, -2

Thus $a_n^{(h)} = C_1(-2)^n + C_2(-1)^n$

To find $a_n^{(p)}$ *:* It is of the form $a_n^{(p)} = A_0 3^n$ and

substitute this in the given relation then we get

$A_0 3^{n+2} + 3A_0 3^{n+1} + 2A_0 3^n = 3^n$

On solving we get $A_0 = 1/20$

Thus $a_n^{(p)} = \frac{1}{20} 3^n$

Hence $a_n = a_n^{(h)} + a_n^{(p)}$

$$= C_1(-2)^n + C_2(-1)^n + \frac{1}{20} 3^n$$

UNIT-VI
Assignment-Cum-Tutorial Questions
Section A:
Objective Questions:

1. Show that the sequence $\{a_n\}$ is a solution of recurrence relation $a_n=-3a_{n-1}+4a_{n-2}$ if $a_n=1$?
2. Find all the solutions of the recurrence relation $a_n = 7a_{n-1} - 12a_{n-2} + 5^n$ ___________________
3. The particular solution of the recurrence relation $a_n = 7a_{n-1} + 8a_{n-2} + (5n+7)7^n$ is of the form ______________________________
4. The particular solution of the recurrence relation $a_n = 13a_{n-1} - 56a_{n-2} + 80a_{n-3} + (3n^2 + 10n + 8)4^n$ is of the form___________________
5. The solution of the recurrence relation $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ with initial condition $a_0 = 1$ $a_1 = 3$ and $a_2 = 7$ is
6. If $r^2 - c_1 r - c_2 = 0$ has only one root r_0 then the general solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ is [ ]

   (a) $a_n = \alpha_1 r_0 - \alpha_2 n r_0$ (b) $a_n = \alpha_1 r_0 + \alpha_2 r_0^n$ (c) $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ (d) None
7. The recurrence relation satisfied by the sequence $a_n = 3n$ can be [ ]

   a) $a_n = a_{n-1} + 5$ b) $a_n = a_{n-1} + 3$ c) $a_n = 2a_{n-1} + 6$ d) $a_n = a_{n-1} + 7$
8. Which of the following is a linear homogenous recurrence relation? [ ]

   a) $a_n = a_{n-1} + 3^n$ b) $a_n = a_{n-1} + 5^n$ c) $a_n = 4a_{n-1} + 4.5^n$ d) all the above
9. Which of the following is a linear homogenous recurrence relation with constant Coefficients [ ]

   a) $a_n = 3a_{n-4}$ b) $a_n = 4a_{n-4} + 5^n$ c) $a_n = 4a_{n-1} + 3a_{n-2}^2$ d) all the above
10. The number of bacteria in a colony doubles in every hour. The recurrence relation for the number of bacteria after n^{th} hours is [ ]

    a) $a_n = 4a_{n-1}$ b) $a_n = 3a_{n-1}$ c) $a_n = 2a_{n-1}$ d) $a_n = 6a_{n-1}$

Section B:
Subjective Questions:

1. Solve the recurrence relation $u_n = 4u_{n-1} - 4u_{n-2} + 2^n$ with $u_0 = 1$, $u_1 = 1$
2. Solve the Recurrence Relation $u_n + 5u_{n-1} + 6u_{n-2} = 3n^2 - 2n + 1$, $u_0=1$, $u_1=1$
3. Solve $na_n + (n-1)a_{n-1} = 2^n$ where $a_0=1$
4. Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$, $n \geq 2$, $a_0=10$, $a_1=41$.
5. Solve the recurrence relation $u_{n+2} - u_{n+1} - 12u_n = 10$, $u_1=13$, $u_0=0$.
6. Solve the recurrence relation $u_{n+2} + 4u_{n+1} + 3u_n = 5(-2)^n$, $u_0=1$, $u_1=0$

7. Find a particular solution for recurrence relation using the method of determined coefficients $a_n - 7a_{n-1} + 12a_{n-2} = 2n$
8. Find a particular solution for recurrence relation using the method of determined coefficients $a_n - 5a_{n-1} = 3^n$?
9. Solve the recurrence relation $a_n - 6a_{n-1} + 8a_{n-2} = 4n$ where $a_0 = 8$ and $a_1 = 22$?

Section C.

Gate Questions:

1.The solution of the recurrence relation $a_n = a_{n-1} + 3$ with initial condition $a_0 = 5$ is

a) 2n+5 b) 3n-5 c) 5n+3 d) 3n+5

2.The characteristic equation of the recurrence relation $a_n = 10a_{n-1} - 16a_{n-2}$ is [ ]

a) 8,2 b) -8,-2 c) 4,6 d)-4,-6

3.The solution for the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ with initial conditions $a_0 = 1$, and $a_1 = 12$ is [ ]

a) $a_n = 5^n + 2n(4^n)$ b) $a_n = 4^n + 6^n$ c) $a_n = 4^n + 2n(4^n)$ d) $a_n = 7^n + 2n(6^n)$

4.Let f_n be the sequence satisfied that $f_n = f_{n-1} + f_{n-2}$, find the explicit formula for f_n with initial conditions $f_0 = 2$, $f_1 = 3$ [ ]

a) $\left(\frac{\sqrt{5}+1}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2}\right)^n$ b) $\left(\frac{\sqrt{5}+1}{2}\right)^n + \left(\frac{-\sqrt{5}+1}{2}\right)^n$ c) $\left(\frac{2}{\sqrt{5}}+1\right)^n + \left(\frac{\sqrt{5}+1}{2}\right)^n$ d) none

5.The recurrence relation $T(n) = 2T(n-1) + n$, $T(1) = 1$, $n \geq 2$ equals to [ ]

a) $2^{n+1} - n - 2$ b) $2^n - n$ c) $2^{n+1} - 2n - 2$ d) $2^n + n$

6.The solution of the recurrence relation $a_n = 4a_{n-1} + 3n$ is [ ]

(a) $a_n = \alpha 4^{n-1} + n + \frac{4}{3}$ (b) $a_n = \alpha 4^n - n - \frac{4}{3}$ (c) $a_n = \alpha 4^{n-1} - n + \frac{4}{3}$ (d) $a_n = \alpha 4^n + n - \frac{4}{3}$