GUDLAVALLERU ENGINEERING COLLEGE

(An Autonomous Institute with Permanent Affiliation to JNTUK, Kakinada)

Seshadri Rao Knowledge Village, Gudlavalleru – 521 356.

Department of Computer Science and Engineering

HANDOUT

on

LA & IT

Vision :

To be a Centre of Excellence in computer science and engineering education and training to meet the challenging needs of the industry and society

Mission:

- To impart quality education through well-designed curriculum in tune with the growing software needs of the industry.
- To be a Centre of Excellence in computer science and engineering education and training to meet the challenging needs of the industry and society.
- To serve our students by inculcating in them problem solving, leadership, teamwork skills and the value of commitment to quality, ethical behavior & respect for others.
- To foster industry-academia relationship for mutual benefit and growth

Program Educational Objectives :

PEO1 : Identify, analyze, formulate and solve Computer Science and Engineering problems both

independently and in a team environment by using the appropriate modern tools.

PEO2 : Manage software projects with significant technical, legal, ethical, social, environmental and

economic considerations.

PEO3 : Demonstrate commitment and progress in lifelong learning, professional development,

leadership and Communicate effectively with professional clients and the public

HANDOUT ON LA & IT

Class & Semester:I B.Tech – I Semester Year:2018-19

Branch: CSE Credits : 4

1. Brief history and current developments in the subject area

"MATHEMATICS IS THE MOTHER OF ALL SCIENCES", It is a necessary avenue to scientific knowledge , which opens new vistas of mental activity. A sound knowledge of engineering mathematics is essential for the Modern Engineering student to reach new heights in life. So students need appropriate concepts, which will drive them in attaining goals.

Importance of mathematics in engineering study :

Mathematics has become more and more important to engineering Science and it is easy to conjecture that this trend will also continue in the future. In fact solving the problems in modern Engineering and Experimental work has become complicated, time – consuming and expensive. Here mathematics offers aid in planning construction, in evaluating experimental data and in reducing the work and cost of finding solutions.

2. Pre-requisites, if any

- Basic Knowledge of Mathematics at Intermediate Level is required.

3. Course objectives:

- To understand the concepts of eigenvalues and eigenvectors.
- To gain the knowledge of Laplace and inverse Laplace transforms.
- To understand the concepts of Fourier Transforms.

4. Course outcomes:

At the end of the course, Students will be able to

CO1: use the concepts of eigenvalues and eigenvectors in Engineering problems.

CO2: apply Laplace transforms to find the solutions of ordinary differential equations.

CO3: find Fourier transforms and inverse transforms for a given function.

5. Program Outcomes:

Graduates of the Computer Science and Engineering Program will have

a) an ability to apply knowledge of mathematics, science, and engineering
b) an ability to design and conduct experiments, as well as to analyze and interpret data
c) an ability to design a system, component, or process to meet desired needs within realistic constraints such as economic, environmental, social, political, ethical, health and safety, manufacturability, and sustainability
d) an ability to function on multidisciplinary teams
e) an ability to identify, formulate, and solve engineering problems
f) an understanding of professional and ethical responsibility
g) an ability to communicate effectively
h) the broad education necessary to understand the impact of engineering solutions in a global, economic, environmental, and societal context
i) a recognition of the need for, and an ability to engage in life-long learning,

j) a knowledge of contemporary issues

6. an ability to use the techniques, skills, and modern engineering tools necessary for engineering practice. **Mapping of Course Outcomes with Program Outcomes:**

	a	B	c	d	E	F	G	h	I	j	K
CO1	M				M						
CO2	M				M						
CO3	M				M						

7. **Prescribed Text books**
   - B.S.Grewal, Higher Engineering Mathematics : 42nd edition, Khanna Publishers,2012 , New Delhi.
   - B.V.Ramana, Higher Engineering Mathematics, Tata-Mc Graw Hill company Ltd..

8. **Reference books**
   - U.M.Swamy, A Text Book of Engineering Mathematics – I & II : 2nd Edition, Excel Books, 2011, New Delhi.
   - Erwin Kreyszig, Advanced Engineering Mathematics : 8th edition, Maitrey Printech Pvt. Ltd, 2009, Noida.
   - Dr. T.K.V.Iyengar, Dr. B.Krishna Gandhi, S.Ranganatham and Dr.M.V.S.S.N.Prasad, Engineering Mathematics, Volume-I : 11th edition, S. Chand Publishers, 2012, New Delhi.

9. **Lecture Schedule / Lesson Plan**

S.No	TOPIC	No of. Periods	No of. Tutorials
	UNIT-I :		
1.	Introduction	1	1
2.	Rank definition	1	
3.	Echelon Form	2	
4.	Normal Form	2	
5.	Solution of Linear system of equations	4	1
6.	LU-Decomposition method	2	
7.	Review and conclusion	1	
	UNIT-II :		
8.	Introduction to Eigen values and Eigen vectors	1	1
9.	Finding Eigen values and Eigen vectors	3	
10.	Properties of Eigen values and Eigen vectors	1	1
11.	Cayley-Hamilton Theorem and problems	3	
12.	Review and conclusion	1	
	UNIT-III :		
13.	Real and complex matrices	1	1
14.	Introduction to Quadratic Form	1	
15.	Nature of Quadratic forms	1	
16.	Canonical form, rank, index, signature of Q.F.	3	
17.	Reducing Q.F to Canonical form by orthogonal transformation	4	
18.	Review and conclusion	1	

	UNIT-IV :		
19.	Laplace transforms of standard functions	1	1
20.	Shifting Theorems	1	
21.	change of scale	1	
22.	Transforms of derivatives	1	
23.	Transforms of integrals	1	1
24.	Unit step function –Dirac's delta function	1	
25.	Evaluation of Improper Integrals	2	
26.	Review and conclusion	1	
	UNIT-V :		
27.	Inverse Laplace transforms	1	1
28.	Inverse Laplace transforms by partial fractions	2	
29.	Convolution theorem (with out proof).	1	
30.	Inverse Laplace transforms by Convolution theorem	2	1
31.	Solutions of ordinary differential equations using Laplace transforms	3	
32.	Review and conclusion	1	
	UNIT-VI :		
33.	Fourier integral theorem & Problems	2	1
34.	Fourier transform, sine and cosine transforms & Problems	3	
35.	Properties of Fourier transform (without proofs)	1	1
36.	Inverse Fourier transforms	2	
37.	Review and conclusion	1	
38.	**Total**	60	12

10. URLs and other e-learning resources

So net CDs &IIT CDs on some of the topics are available in the Digital Library.

11. Digital Learning Materials:

- http://nptel.ac.in/courses/106106094
- http://nptel.ac.in/courses/106106094/40
- http://nptel.ac.in/courses/106106094/30
- http://nptel.ac.in/courses/106106094/32
- http://textofvideo. nptl.iitm.ac.in/106106094/lecl.pdf

12. Seminars / group discussions, if any and their schedule: Nil

LINEAR ALGEBRA & INTEGRAL TRANSFORMS
UNIT-I
SYSTEM OF LINEAR EQUATIONS

Objectives:

- To introduce the concept of rank of a matrix.
- To know methods of solving system of Linear equations.
- To be familiar with LU-Decomposition method.

Syllabus:

Rank of a matrix-Echelon form, Normal form, system of equations-Consistence and inconsistence, solving non-homogeneous system of equations by LU-Decomposition.

Learning Outcomes:

Students will be able to

- Calculate rank of a matrix.
- Solve system of Linear equations using by LU-Decomposition
- find an LU decomposition of simple matrices and apply it to solve systems of equations , be aware of when an LU decomposition is unavailable and when it is possible to circumvent the problem

UNIT – I
LEARNING MATERIAL

Introduction of Matrices:

Definition :

A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote matrices by capital letters as *A,B, C* etc

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \left(a_{ij}\right)_{m \times n}$$

Remark: A matrix is not just a collection of elements but every element has assigned a definite position in a particular row and column.

Sub – Matrix: Any matrix obtained by deleting some rows or columns or both of a given
matrix is called sub matrix.

Minor of a Matrix: let A be an mxn matrix. The determinant of a square sub matrix of A is called a minor of the matrix.

Note: If the order of the square sub matrix is 't' then its determinant is called a minor of order 't'.

Rank of a Matrix:
Definition:
A matrix is said to be of rank r if

i. It has at least one non-zero minor of order r and
ii. Every minor of order higher than r vanishes.

Rank of a matrix A is denoted by $\rho(A)$.

Properties:

1) The rank of a null matrix is zero.
2) For a non-zero matrix A , $\rho(A) \geq 1$
3) The rank of every non-singular matrix of order n is n. The rank of a singular matrix of order n is < n.
4) The rank of a unit matrix of order n is n.
5) The rank of an m × n matrix $\leq \min(m, n)$.
6) The rank of a matrix every element of which is unity is one
7) Equivalent matrices have the same order and same rank because elementary transformation do not alter its order and rank.
8) Rank of a matrix is unique.
9) Every matrix will have a rank

Elementary Transformations on a Matrix:

i) Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$
ii) If i^{th} row is multiplied with k then it is denoted by $R_i \to k\,R_i$
iii) If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then if is denoted by $R_i \to R_i + kR_j$.

Note: 1. The corresponding column transformations will be denoted by writing 'c'

$$i.e\; c_i \leftrightarrow c_j,\quad c_i \leftrightarrow k\,c_j\;,\quad c_i \to c_i + kc_j$$

2. The elementary operations on a matrix do not change its rank.

Equivalence of Matrices: If B is obtained from A after a finite chain of elementary transformations then B is said to be equivalent to A. It is denoted as B~A.

Different methods to find the rank of a matrix:
Method 1:
Echelon form: A matrix is said to be in Echelon form if
1) Zero rows, if any, are below any non-zero row
2) The number of zeros before the first non-zero elements in a row is less than the number of such zeros in the next rows.

Ex: The rank of matrix which is in Echelon form $\begin{bmatrix} 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is 3 since the no. of non-zero rows is 3

Note:1. Apply only row operations while reducing the matrix to echelon form

Problem: Find the rank of the matrix A = $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ by reducing into echelon form

Sol: A = $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

R_3-3R_1, R_4-R_1

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

R_3-R_2, R_4-R_2

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in echelon form

Rank = no. of non zero rows = 2

Method 2:

<u>**Normal Form**</u>: Every m×n matrix of rank r can be reduced to the form of $I_r, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, [I_r \quad 0], \begin{bmatrix} I_r \\ 0 \end{bmatrix}$ by a finite chain of elementary row or column operations where I_r is the Identity matrix of matrix of order r.

Normal form another name is “canonical form”

Problem: Find the rank of matrix A = $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ by reducing it to canonical form

Sol: Given matrix A = $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix} R_2-4R_1, R_3-2R_1, R_4-R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} C_2+C_1, C_4-3C$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \end{bmatrix} (-1)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \end{bmatrix} R_4-6R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -16 \end{bmatrix} C_3+C_2, C_4-C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -16 \end{bmatrix} \frac{1}{6}C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -16 \end{bmatrix} \quad C_4+16C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$$

The above matrix is in normal form and rank is 3.

<u>Elementary matrix:</u>

A matrix obtained from a unit matrix by a single elementary transformation.

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

<u>Linear Equation:</u> An Equation is of the form $a_1x_1+a_2x_2+a_3x_3+\ldots..+\ldots.a_nx_n = b$ where $x_1,x_2,----,x_n$ are unknown and $a_1,a_2,\ldots a_n$, b are constants is called a linear equation in 'n' unknowns.

<u>Consistency of System of Linear equations (Homogeneous and Non Homogeneous) Using Rank of the Matrix:</u>

A System of m linear algebraic equations in n unknowns $x_1,x_2,x_3,\ldots\ldots..x_n$ is a set of equations of the form

$$\left.\begin{array}{llll} a_{11}x_1 + & a_{12}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + & & a_{1n}x_n = b_1 \\ a_{21}x_1 + & a_{22}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + & & a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\ a_{m1}x_1 + & a_{m2}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +. & & a_{mn}x_n = b_m \end{array}\right\} \rightarrow (1)$$

The numbers a_{ij}'s are known as coefficients and b_i are known as constants of the system (1) can be expressed as $\sum_{j=i}^{n} a_{ij}x_j$ = b_i ,i = 1,2,….m

Non homogenous System **:** When at least one b_i is nonzero .

Homogenous System: If $b_i = 0$ for i = 1,2, ...m (all R.H.S constants are zero) Solution of system (1) is set of numbers $x_1, x_2, x_3, \ldots\ldots x_n$ which satisfy simultaneously all the equations of the system (1)

Trivial Solution is a solution where all x_i are zero i.e $x_1 = x_2 = \ldots\ldots = x_n = 0$
The set of equations can be written in matrix form as AX = B → (2)

Where A = $\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots\ldots\ldots & & \\ a_{m1} & a_{m2} & \ldots\ldots\ldots & a_{mn} \end{bmatrix}$ is called the coefficient matrix

X = $\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ is the set of unknowns B = $\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$ is a column matrix of constants

Consistent : A system of equations is said to be consistent if (1) has at least one solution.
Inconsistent if system has no solution at all
Augmented matrix [A B] of system (1) is obtained by augmenting A by the column B
Matrix equation for the homogenous system of equations is AX = 0 ....(3)
It is always consistent.
If X_1, X_2 are two solutions of equation (3) then their linear combination $k_1x_1 + k_2x_2$ where k_1 & k_2 are any arbitrary numbers, is also solution of (3) .The no. of L.I solutions of m homogenous linear equations in n variables , AX = 0 , is (n – r) where r is the rank of the matrix A.

Nature of solution:

non-homogeneous with m equations and n unknowns
The system of equations AX=B is said to be

i. consistent and unique solution if rank of A = rank of [A|B] = n i.e.,r = n
   Where r is the rank of A and n is the no. of unknowns.
ii. Consistent and an infinite no. of solutions if rank of A= rank of [A|B] < n i.e., r < n. In this case we have to give arbitrary values to n-r variables and the remaining variables can be expressed in terms of these arbitrary values.
iii. Inconsistent if rank of A ≠ rank of [A|B]

Note: Method of finding the rank of A and [A|B]:
Reduce the augmented matrix [A:B] to Echelon form by elementary row transformations.

Problem: Show that the equations x+y+z = 4, 2x+5y-2z =3, x+7y-7z =5 are not consistent.

Sol: write given equations is of the form AX = B

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

Consider augmented matrix $\left[A/B\right] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

$$Applying\ R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$$

$$Applying\ R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

Thus $\rho(A) = 2\ and\ \rho\left[A/B\right] = 3$

Therefore $\rho(A) \neq \rho\left[A/B\right]$

Hence the given system is an inconsistent.

Homogeneous linear equations:

Consider the system of m homogeneous equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \text{..........} + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \text{..........} + a_{2n}x_n &= 0 \\ \text{..} \\ a_{m1}x_1 + a_{n2}x_2 + \text{..........} + a_{mn}x_n &= 0 \end{aligned} \qquad (1)$$

(1) can be written as AX=O

Where A is the coefficient matrix formed by A = $\begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix}$

X= $\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ and B= $\begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$

Consistency: The matrix A and [A|B] are same. So rank of A = rank of [A|B]
Therefore the system (1) is always consistent.

Nature of solution:

Trivial solution: Obviously $x_1=x_2=x_3=$ -------- $=x_n=0$ is always a solution of the given
system and this solution is called trivial solution.
Therefore trivial solution or zero solution always exists.

Non-Trivial solution: Let r be the rank of the matrix A and n be the no. of unknowns.

Case-I: If r=n, the equations AX=O will have n-n i.e., no linearly independent solutions. In this case, the zero solution will be the only solution.

Case-II: If r<n, we shall have n-r linearly independent solutions. Any linear combination of
these n-r solutions will also be a solution of AX=O.

Case-III: If m<n then r≤m<n. Thus in this case n-r > 0.
Therefore when the no. of equations < No. of unknowns, the equations will have an infinite no. of solutions.

Note: The system Ax =0 possesses a non-zero solution if and only if A is a singular matrix.

Introduction of LU Decomposition :

A m×n matrix is said to have a **LU-decomposition** if there exists matrices L and U with the following
properties:

(i) L is a m×n lower triangular matrix with all diagonal entries being 1.
(ii) U is a m×n matrix in some echelon form.
(iii) *A* = *LU*.

Procedure to solve by LU Decomposition:

Suppose we want to solve a m×n system $A\boldsymbol{X} = \boldsymbol{b}$.
If we can find a LU-decomposition for A , then to solve $A\boldsymbol{X} = \boldsymbol{b}$, it is enough to solve the systems

$$\left.\begin{array}{l} LY = b \\ UX = Y \end{array}\right\}.$$

Thus the system LY = b can be solved by the method of forward substitution and the system *UX* = ***Y***
can be solved by the method of backward substitution. To illustrate, we give some examples

It turns out that we need only consider lower triangular matrices L that have 1's down the diagonal. Here is an example, let A = $\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}$ = LU where L = $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and U = $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ Multiplying out LU **and** setting the answer equal to A gives $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \quad l \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$. Now we have to use this to find the entries in L and U. Fortunately this is not nearly as hard as it might at first seem. We begin by running along the top row to see that u_{11} = 1 , u_{12} =5 , u_{13} = 1 . Now consider the second row $l_{21}u_{11}$ = 2 l_{21} × 1=2 l_{21} = 2 , $l_{21}u_{12}$ + u_{22} = 1 2 × 5 +u_{22} = 1 u_{22} =-9 , $l_{21}u_{13}$ + u_{23} = 3 2 × 1 + u_{23} = 3 u_{23} =1 .
Now we solve the system LY=B i.e.,
$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 14/9 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 16 \end{bmatrix}$ by forward substitution y_1=9,y_2=-6,y_3=-5/3

And the system UX=Y i.e., $\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ -5/3 \end{bmatrix}$ by backward substitution x=1,y=1,z=3.

UNIT-I
Assignment-cum-Tutorial Questions
SECTION-A

Objective Questions

1. The rank of I_3 = __________
2. The rank of $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix}$ is __________
3. If the rank of a matrix is 4. Then the rank of its transpose is________
4. The rank of a matrix in echelon form is equal to _______
5. The necessary and sufficient condition that the system of equations AX=B is consistent if_______________
6. The value of K for which the system of equations 5x+3y=12, 15x+9y=k-3 has infinitely many solution is __________
7. The non trivial solution of system of equations $2x - 3y = 0$ *and* $-4x + 6y = 0$ *is* _____
8. The system of equations $x + y + z = 3, x + 2y + 3z = 4, x + 4y + 9z = 6$ will have _____
9. If the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 5 \\ 2 & k & 4 \end{bmatrix}$ is 2 then k=_______

10.The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ [ ]

(a) 0 (b) 1 (c) 2 (d) 3

11. If 5 non homogeneous equations are given with 4 unknowns. The system of equations AX=B consistent if [ ]

(a) The rank of A=4 (b) the rank of A is 3
(c) the rank of A <4 (d) the rank of A is 5

12.If the system of equations $x - 3y - 8z = 0, 3x + y - \lambda z = 0, 2x + 3y + 6z = 0$

possess a nontrivial solution then λ = [ ]

(a) 2 (b) $\frac{-4}{9}$ (c) 6 (d) 8

13. Every square matrix can be written as a product of lower and upper

triangular matrices if [ ]

(a)atleast one principal minor is zero (b) all principal minors are non-zero
(c) all principal minors are zero (d) atleast one principal minor is non-zero

14. Consider two statements: [ ]

i. P: Every matrix has rank
ii. Q: Rank of a matrix is not unique

(a) Both P and Q are false (b) Both P and Q are true

(c) P is true and Q is false (d) P is false and Q is true

15. Which of the following statement is correct [ ]

a. Rank of a Non-zero matrix is Zero

b.Rank of a rectangular matrix of order mxn is m when m > n

c.Rank of a rectangular matrix of order mxn is m when m < n

d.Rank of a square matrix of order nxn is n+1.

16. Rank of a non singular matrix of order m is [ ]

a. m b. n c. 0 d. not defined

17. Rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is [ ]

a. 1 b. 2 c. 3 d. 4

18. Find the values of k_1 and k_2 for which the non-homogeneous linear system, [ ]

$3x - 2y + z = k_2$; $5x - 8y + 9z = 3$; $2x + y + k_1 z = -1$ has no solution

a) $k_1 = -3$, $k_2 = 1/3$ b) $k_1 = 3$, $k_2 \neq 1/3$

c) $k_1 = -3$, $k_2 \neq 1/3$ d) $k_1 = 3$, $k_2 = 1/3$

19. The equations x + 4y + 8z = 16, 3x + 2y + 4z = 12 and 4x + y + 2z = 10 have [ ]

a) only one solution b) two solutions

c) infinitely many solutions d) no solutions

SECTION-B

Subjective Questions :

1. Determine the rank of matrix by reducing to echelon form

i) $A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix}$ ii) $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

iii) $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ $vi)$ $A = \begin{bmatrix} 3 & -1 & 2 & 1 \\ 1 & 4 & 6 & 1 \\ 7 & -11 & -6 & 1 \\ 7 & 2 & 12 & 3 \end{bmatrix}$

2. Find the rank of the following matrices by reducing them into Normal form.

a) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 10 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

3. Find the rank of the following matrices by reducing them into Canonical form

$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$, $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

4. Test for the consistency and solve the following equations: 2x-3y+7z =5; 3x+y-2z =13; 2x +19y -47z =32
5. Investigate for what values of a and b the simultaneous equations x +a y + z = 3; x + 2y +2z = b; x +5y +3z =9 have
   a) no solution b) a unique solution c) infinitely many solutions
6. Test for consistency and solve if the equations are consistent $x+2y+2z=2, 3x-y+3z=-4, x+4y+6z=0$
7. Solve the system of equations by using LU Decomposition method 3x+2y+2z=4, 2x+3y+z=5, 3x+4y+z=7.
8. Express A = $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$ as a product of LU. .
9. Test for the consistency of following and solve the following equations: x +2y +z =3 ; 2x +3y+2z =5; 3x -5y +5z =2; 3x +9y –z =4
10. For what value of k the equations x + y + z =1;2x + y + 4z = k;4x + y + 10z=k^2 have a solution and solve them completely in each case.

SECTION-C

GATE Previous Paper Questions

1. The system of linear equations $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5` \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 14 \end{bmatrix}$ has [ ]

   (GATE 2014)

   a) A unique solution b) infinitely many solutions
   b) No solution d) exactly two solutions
2. The system of equations x + y + z =6, x + 4y + 6z = 20, x + 4y + λz = u **(GATE 2011)**
   has no solution for values of λ and μ given by [ ]

a) $\lambda = 6, \mu = 20$ b) $\lambda = 6, \mu \neq 20$

c) $\lambda \neq 6, \mu = 20$ d) $\lambda \neq 6, \mu \neq 20$

3. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is **(GATE 2006)**

a) 0 b) 1 c) 2 d) 3 [ ]

4. The determinant of a matrix A is 5 and the determinant of matrix B is 40 the determinant of matrix AB is____________ **(GATE2014)**

5. Consider the following system of equations 3x + 2y = 1, 4x + 7z = 1, x + y + z =3, x- 2y+7z=0 The number of solution for this system is **(GATE 2014)**

6. The following system of equations $x_1+x_2+2x_3 = 1$, $x_1 + 2x_2+3x_3 = 2$, $x_1+4x_2+\alpha x_3 = 4$ has α unique solution the only possible values of α are **(GATE2008)**

a) 0 b) either 0 or 1 c) one of 0,1, or -1 d) any real number

7. Consider the following system of equations in three variables x_1,x_2 and x_3 $2x_1-x_2+3x_3=1$, $3x_1+2x_2+5x_3=2$, $-x_1+4x_2+x_3=3$ then The system of equations has **(GATE 2005)**

a) No Solutions b) More than one but a finite number of solutions

c) Unique solutions d) All infinite number of solutions

8. How many solutions does the following system of linear equations have –x + 5y = -1, x + 3y =3 , x – y =2 **(GATE 2013)**

a) Infinitely many b) Two distinct solutions

c) Unique d) None [ ]

9. For matrices of same dimension M and N and a scalar C which of these properties does not always hold **(GATE 2014)**

a) $\left(M^T\right)^T = M$ b) $(CM)^T = CM^T$

c) $(M+N)^T = M^T + N^T$ d) MN = NM [ ]

10. In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal elements of U are both 1,then lower diagonal entry l_{22} of L is __________. **(GATE 2009)**

a) 4 b) 5 c) 6 d) 7 [ ]

LINEAR ALGEBRA & INTEGRAL TRANSFORMS

UNIT-II

EIGEN VALUES AND EIGEN VECTORS

Objectives:

- To understand eigen values, eigen vectors and their properties, cayley Hamilton theorem.

Syllabus:

Eigen values and eigen vectors, Properties of eigen values and eigen vectors (with out proof), Cayley-Hamilton theorem (with out proof)- Finding inverse and power of a matrix .

Course Outcomes:

Students will be able to

- Find eigen values and eigen vectors of a matrix
- Apply Cayley-Hamilton Theorem to compute powers and inverse of a given square matrix

UNIT – II

LEARNING MATERIAL

Eigen values and eigen vectors of a matrix:

Consider the following 'n' homogeneous equations in 'n' unknowns as given below

$$(a_{11} - \lambda)\, x_1 + a_{12}x_2 + - - - - - - a_{1n}x_n = 0$$

$$a_{21}x_1 + (a_{22} - \lambda)\, x2 + - - - - - -a_{2n}x_n = 0$$

…………………………………….

$$a_{n1}x_1 + a_{n2}x_2 + - - - - - - (a_{nn} - \lambda)x_n = 0$$

The above system of equations in matrix notation can be written as $(A - \lambda I)\, X = 0$

Where 'λ' is a parameter.

The matrix $(A - \lambda I)$ is called 'Characteristic Matrix' and $|A - \lambda I| = 0$ is called 'Characteristic Equation' of A . i.e.,

$$|A - \lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + - - - - - - + k_n = 0$$

Where $k_1, k_2, - - - - , k_n$ are expressible in terms of the elements a_{ij}

Eigen Value: The roots of characteristic equation are called the characteristic roots or latent roots or eigen values.

Eigen Vector: If λ is a characteristic root of a matrix then a non-zero vector X such that $AX = \lambda X$ is called a characteristic vector or Eigen vector of A corresponding to the characteristic root λ.

Note: (i) Eigen vector must be a non-zero vector

(ii) Eigen vector corresponding to a eigen value need not be unique

PROPERTIES OF THE EIGEN VALUES:

- The sum of the Eigen values of the square matrix is equal to its trace and product of the Eigen values is equal to its determinant.
- If λ is an eigen value of A corresponding to the eigen vector X then λ^n is the eigen value of the matrix A^n corresponding to the eigen vector X .
- If $\lambda_1, \lambda_2, \lambda_3$, ----, λ_n. are the latent roots of A then A^3 has the latent roots as λ^3_1, λ^3_2 , λ^3_3, ----, λ^3_n.
- A square matrix A and its transpose A^T have the same eigen values.
- If A and B are n rowed square matrix and if A is invertible then $A^{-1}B$ and BA^{-1} have the same eigen values.
- If $\lambda_1, \lambda_2, \lambda_3$, ----, λ_n. are the eigen values of a matrix A then $k\lambda_1, k\lambda_2, k\lambda_3$, ----, $k\lambda_n$. are the eigen values of the matrix KA where K is a non-zero scalar.
- If λ is an Eigen value of the matrix A then $\lambda+k$ is an Eigen value of the matrix A+KI.
- If λ is the Eigen value of A then λ-K are the eigen values of the matrix A-KI.
- If λ_1, λ_2 , λ_3 , ----, λ_n. are the eigen values of a matrix A then $(\lambda_1 - \lambda)^2$, $(\lambda_2 - \lambda)^2$,............. $(\lambda n - \lambda)^2$ are the eigern values of the matrix $(A- \lambda I)^2$.
- If λ is an Eigen value of a non-singular matrix A then λ^{-1} is an Eigen value of the matrix A^{-1} corresponding to the eigen vector X .
- If λ is an Eigen value of a non-singular matrix A then $|A| / \lambda$ is an eigen value of the matrix adjA.
- If λ is an Eigen value of a non-singular matrix A then $1/\lambda$ is an eigen value of A^{-1}.
- If λ is an Eigen value of a non-singular matrix A then the eigen value of $B = a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$.
- The eigen values of a triangular matrix are just diagonal elements of the matrix.
- If A and B are non-singular matrices of same order,then AB and BA have the same eigen values.

- Suppose A and P are square matrices of order n such that P is non-singular,then A and $P^{-1}AP$ have the same eigen values.
- The eigen values of real symmetric matrix are real.
- For a real symmetric matrix ,the eigen vectors corresponding to two distinct eigen values are orthogonal.
- The two eigen vectors corresponding to two different eigen values are linearly independent.

Finding Eigen vectors:

Method1:

Case(i): Eigen values are distinct $\lambda_1 \neq \lambda_2 \neq \lambda_3$ (suppose the matrix A of order 3)

Corresponding to the eigen value λ_1, the eigen vector $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ can be obtained from the matrix equation $(A-\lambda_1)X_1=O$ and by expanding it we get three homogeneous linearly independent equations are obtained and solving any two equations for x_1,x_2,x_3 the eigen vector

$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ can be obtained. Similarly, the remaining Eigen vectors X_2, X_3 can be obtained corresponding to the Eigen values λ_2 and λ_3.

Case(ii): Finding Linearly Independent Eigen vectors of a matrix when the Eigen values of the matrix are repeated ($\lambda_1=\lambda_2$)

The matrix equation $(A-\lambda I)\,X=O$ gives three equations which represent a single independent equation of the form.

$a_1x_1+a_2x_2+a_3x_3 = 0$

We have to choose two unknowns as $k_1,k_2.$

So we can get two linearly independent Eigen vectors X_1 and X_2

Method2: (Rank method) in the matrix equation $(A-\lambda I)\,X=O$, reduce the coefficient matrix to Echelon form, the rank of the coefficient matrix is less than the number of unknowns. So give arbitrary constants to (n-r) variables and solve as in case of homogeneous equations.

Example 1: Find the Eigen values and corresponding Eigen vectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol: The characteristic equation of the matrix A is $|A-\lambda I| = 0 \Rightarrow$
$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - 10\lambda^2 + 20\lambda - 32 = 0 \Rightarrow (\lambda-2)(\lambda-2)(\lambda-8) = 0 \Rightarrow \lambda = 2, 2, 8$

The Eigen values of A are 2, 2 and 8.

Let the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to the Eigen value λ is given by the non-zero

solution of the equation $(A-\lambda I) X = O$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\lambda=8$, then the Eigen vector X_1 is given by $(A-8I) X_1 = O$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x_1-2x_2+2x_3 = 0, -2x_1-5x_2-x_3 = 0, 2x_1-x_2-5x_3 = 0$

Solving any two of the equations, we get $\Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} = k$ (say)

$\Rightarrow x_1=2k,\ x_2= -k,\ x_3=k$ (k is arbitrary) $\Rightarrow X_1= \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} \Rightarrow X_1 = k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

The eigen vector corresponding to $\lambda_1=8$ is $X_1= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

If $\lambda=2$, then the Eigen vector X_2 is given by $(A-2I) X_2 = O$

$$\Rightarrow \begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $4x_1-2x_2+2x_3 = 0$, $-2x_1+x_2-x_3 = 0$, $2x_1-x_2+x_3 = 0$ $\Rightarrow$ $2x_1-x_2+x_3 = 0$

Let $x_2 = k_2$, $x_3 = k_1$, $2x_1 = k_2-k_1$ $\therefore$ $X_2 = \begin{bmatrix} \frac{k_2-k_1}{2} \\ k_2 \\ k_1 \end{bmatrix} = 2\begin{bmatrix} \frac{k_2-k_1}{} \\ 2k_2 \\ 2\,k_1 \end{bmatrix} = 2k_1\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + 2k_2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\therefore$ Eigen vectors corresponding to $\lambda = 2$ are $X_2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\therefore$ Eigen vectors of A are $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Cayley-Hamilton theorem:

- *Every square matrix satisfies its own characteristic equation.*

Remark: (i) Determination of A^{-1} using Cayley-Hamilton theorem

Let A be n-rowed square matrix.By Cayley-Hamilton theorem ,A satisfies its own characteristic equation. i.e $(-1)^n[A^n+a_1A^{n-1}+a_2A^{n-2}+\ldots\ldots+a_nI] = 0$

$\Rightarrow A^n +a_1A^{n-1}+a_2A^{n-2}+\ldots\ldots+a_nI = 0$ ------ (1)

$\Rightarrow A^{-1}[\ A^n +a_1A^{n-1}+a_2A^{n-2}+\ldots\ldots+a_nI] = 0$

If A is a non-singular ,then we have $a_nA^{-1} = -A^{n-1}-a_1A^{n-2}-\ldots\ldots-a_{n-1}I$

$\Rightarrow A^{-1} = \left(\frac{-1}{a_n}\right)[A^{n-1}+a_1A^{n-2}-\ldots\ldots+a_{n-1}I]$

Remark: (ii) Determination of powers of **A** using Cayley-Hamilton theorem

Multiplying equation (1) with A, we get $A^{n+1} +a_1A^n+a_2A^{n-1}+\ldots\ldots+a_nA = 0$

$\Rightarrow A^{n+1} = -[a_1A^n+a_2A^{n-1}+\ldots\ldots+a_nA]$

Example 1: If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ verify Cayley-Hamilton theorem .Find A^4 and A^{-1} using Cayley-Hamilton theorem.

Solution: Given matrix is $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(\lambda^2 - 2\lambda - 3) - 2(6-2\lambda) - 1(-6+2\lambda) = 0$$

$\Rightarrow -\lambda^3 + 3\lambda^2 + 3\lambda - 9 = 0 \Rightarrow \lambda^3 - 3\lambda^2 - 3\lambda + 9 = 0$ ------(i)

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{-2} & \mathbf{1} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{-2} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{-2} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \mathbf{6} & \mathbf{-6} \\ \mathbf{0} & \mathbf{9} & \mathbf{-6} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{-2} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{3} & \mathbf{6} & \mathbf{-6} \\ \mathbf{0} & \mathbf{9} & \mathbf{-6} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \mathbf{24} & \mathbf{-21} \\ \mathbf{6} & \mathbf{21} & \mathbf{-24} \\ \mathbf{6} & \mathbf{-6} & \mathbf{3} \end{bmatrix}$$

Consider $A^3 - 3A^2 - 3A + 9I = \begin{bmatrix} \mathbf{3} & \mathbf{24} & \mathbf{-21} \\ \mathbf{6} & \mathbf{21} & \mathbf{-24} \\ \mathbf{6} & \mathbf{-6} & \mathbf{3} \end{bmatrix} - 3\begin{bmatrix} \mathbf{3} & \mathbf{6} & \mathbf{-6} \\ \mathbf{0} & \mathbf{9} & \mathbf{-6} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix} - 3\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{-2} & \mathbf{1} \end{bmatrix} + 9\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} = O$

$A^3 - 3A^2 - 3A + 9I = O$ -------- (ii)

Matrix A satisfies its own characteristic equation

Cayley-Hamilton theorem is verified by A

To find A^{-1} :Multiplying equation (ii) with A^{-1} on both sides

$A^{-1}[\ A^3 - 3A^2 - 3A + 9I] = A^{-1}(\ O) \Rightarrow A^2 - 3A - 3I + 9A^{-1} = 0$

$$\Rightarrow 9A^{-1} = 3A + 3I - A^2 = 3\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 6 & -3 & 0 \\ 6 & -6 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{9}\begin{bmatrix} 3 & 0 & 3 \\ 6 & -3 & 0 \\ 6 & -6 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

To find A^4 : Multiplying equation (ii) with A on both sides

$A[\ A^3 - 3A^2 - 3A + 9I] = A(\ O) \Rightarrow A^4 - 3A^3 - 3A^2 + 9A = O$

$$\Rightarrow \quad A^4 = \quad 3A^3+3A^2-9A \quad = \quad 3\begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} + 3\begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} - 9\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 72 & -72 \\ 0 & 81 & -72 \\ 0 & 0 & 9 \end{bmatrix}$$

UNIT-II

Assignment-cum-Tutorial Questions

Section A

Objective Questions

1. Two of the eigen values of a 3×3 matrix whose determinant equals 4 are -1 and 2 then the third eigen value of the matrix is equal to____________________
2. The Eigen values of A= $\begin{bmatrix} 1 & 0 & -0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are ___________________________
3. If the Eigen values of A are 1,3,0 then $|A|$ = _______________
4. The Eigen values of A are (1,-1,2) then the eigen values of Adj(A) are ________________
5. If one of eigen values of A is 0 then A is _____________
6. The eigen value of adj A is _______________
7. If A is orthogonal then A^{-1}= ___________
8. Can an eigen vector be a zero vector?(yes/ no)
9. The eigen values of A^2 are ____________where A=$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$
10. Can a zero value be an eigen value?(yes/ no)
11. If 2,1,3 are the eigen values of A then the eigen values of B=3A+2I are _____________
12. If A is a singular matrix then ______________is an eigen value.
13. Identify the relation between geometric and algebraic multiplicity.
14. The sum of two eigen values and trace of a 3×3 matrix are equal then the value of $|A|$ is___________________
15. Compute characteristic equation of A= $\begin{bmatrix} 3 & -2 & -8 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.
16. The matrix A has eigen values $\lambda_i \neq 0$ then A^{-1}-2I+A has eigen values --------------

19.The Eigen values of A are 2,3,4 then the Eigen values of 3A are __ [ ]

(a)2,3,4 (b) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ (c) -2,3,2 (d) 6,9,12

20.If A =$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then A^3 = [ ]

(a) $2A^2 + 5A$ (b) $4A^2 + 2A$
(c) $2A^2 + 5A$ (d) $5A^2 + 2A$

Section B

Descriptive Questions :

1. Find the eigen values and eigen vectors of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

2. Obtain the latent roots and latent vectors of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

3. Find the eigen values and eigen vectors of $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$

4. Find the characteristic values and characteristic vectors of $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

5. Verify that sum of eigen values is equalto trace of A for A= $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and find the corresponding eigen vector.

6. Verify Cayley Hamilton theorem for A= $\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ Hence find A^{-1} and A^{4}

7. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$. Hence find A^{-1} and A^{4}

8. For the matrix A = $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $3A^3+5A^2-6A+2I$.

9. For the matrix A= $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}$ Find the eigen values and eigen vectors of A^{-1}

10. Using Cayley Hamilton theorem find A^4 for the matrix A= $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Section C

GATE Previous Paper Questions:

1. Eigen vector of the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$ is **(GATE-2004)** [ ]

a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ d) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

2. For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ the eigen value corresponding to eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is **(GATE-2006)** [ ]

a) 2 b) 4 c) 6 d) 8

3. The eigen value of the matrix $\begin{bmatrix} 5 & 3 \\ 3 & -3 \end{bmatrix}$ is **(GATE-1999)** [ ]

a)6 b)5 c) -3 d)-4

4. The 3 characteristic roots of A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ are (**GATE-2000)** [ ]

a)2,3,3 b)1,2,2 c)1,0,0 d)0,2,3

5. The sum of the eigen values of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are (**GATE-2004)** [ ]

a)5 b)7 c)9 d)18

6. Eigen values of S=$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 and 1.Eigen values of S^2=SS are[ ] **(GATE-2006)**

a)1,25 b)6,4 c)5,1 d)2,10

7. One of the eigen vectors of A=$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ is **(GATE-2010)** [ ]

a) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ c) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The minimum and maximum eigen value of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are -2, 6.what is other eigen value? **(GATE-2007)** [ ]

a)5 b)3 c)1 d)-1

8. All the four entries of 2x2 matrix P=$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ are non-zero and one of its eigen value is zero which of the following is true? **(GATE-2008)** **[ ]**

a)$p_{11}p_{22}-p_{12}p_{21}=1$ b)$p_{11}p_{22}-p_{12}p_{21}=-1$
c)$p_{11}p_{22}-p_{12}p_{21}=0$ d)$p_{11}p_{22}+p_{12}p_{21}=0$

9. Eigen values and the corresponding eigen vectors of a 2x2 matrix are given by

Eigen value	**Eigenvector**
$\lambda = 8$	X=$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\mu = 4$	Y=$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then the matrix is **(GATE-2006)** [ ]

a)$\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ b)$\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$ c)$\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ d)$\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

10. The characteristic equation of A is $t^2-t-1=0$, then [ ]

(GATE-2000)

a) A^{-1} does not exist
b) A^{-1} exist but cannot be determined from the data
c) $A^{-1}=A+I$ d) $A^{-1}=A-I$

11. A particular 3x3 matrix has an eigen value -1.The matrix A+I reduces to $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,corresponding to eigen value -1 ,all eigen vectors of A are non-zero vectors of the form [ ] **(GATE-2002)**

a) $\begin{bmatrix} 2t \\ 0 \\ t \end{bmatrix}$,$t \in R$ b) $\begin{bmatrix} 2t \\ s \\ t \end{bmatrix} s,t \in R$ c) $\begin{bmatrix} t \\ 0 \\ -2t \end{bmatrix}$ $t \in R$ d) $\begin{bmatrix} t \\ s \\ 2t \end{bmatrix}$ $s,t \in R$

12. By Cayley-hamilton theorem A = $\begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ satisfies the relation [ ]

(GATE-2007)

a)$A+3I+2A^2=0$ b)$A^2+2A+2I=0$ c) $(A+I)(A+2I)=0$ d)$\exp(A)=0$

14. From question (13), $A^9 =$ [ ]

a)$511A+510I$ b)$309A+104I$ c)$154A+155I$ d)$\exp(9A)$

15. The number of linearly independent eigen vectors of $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is [ ]

(GATE-2007)

a) 0 b)1 c)2 d)infinite

UNIT-III
Quadratic forms

Objectives:

- To understand quadratic forms and their nature.

Syllabus: Real and complex matrices, Introduction to Quadratic Form, Canonical Form – Index, Signature and nature. Reduction of Quadratic forms to canonical forms by orthogonal transformation.

Course Outcomes:

Students will be able to

- Reduce a quadratic form to canonical form.
- Determine index, signature, rank and nature of a quadratic form.

Real Matrix: A real matrix is a matrix whose elements consist entirely of real numbers.

Complex Matrix: A complex matrix is a matrix whose elements may contain complex numbers.

Conjugate Matrix: Suppose A is any matrix, then the conjugate of the matrix A is denoted by $\overline{\mathrm{A}}$ and is defined as the matrix obtained by taking the conjugate of every element of A.

- Conjugate of $a+ib$ is $a-ib$
- $\overline{\overline{\mathrm{A}}}=\mathrm{A}$
- $\overline{\mathrm{A}+\mathrm{B}}=\overline{\mathrm{A}}+\overline{\mathrm{B}}$
- $\overline{\mathrm{AB}}=\overline{\mathrm{A}}\overline{\mathrm{B}}$

Conjugate Transpose of a matrix (or) Transpose conjugate of a matrix:
Suppose A is any

square matrix, then the transpose of the conjugate of A is called Transpose conjugate of A .

And it is denoted by A^{θ}.

$$i.e\ \mathrm{A}^{\theta}=(\overline{\mathrm{A}})^{\mathrm{T}}=\overline{(\mathrm{A}^{\mathrm{T}})}$$

- $(A^{\theta})^{\theta}=A$
- $(A+B)^{\theta}=A^{\theta}+B^{\theta}$
- $(AB)^{\theta}=B^{\theta}A^{\theta}$

Hermitian Matrix: A square matrix is said to be Hermition if $A^{\theta} = A$

- The diagonal elements of Hermitian matrix are purely Real numbers.
- The eigen values of Hermitian matrix are real.

Skew Hermitian Matrix: A square matrix is said to be Skew Hermitian if $A^{\theta} = -A$

- The diagonal elements of Skew Hermitian matrix are either '0' or Purely Imaginary.
- The eigen values of skew hermitian matrix are purely imaginary or zero.

Orthogonal Matrix: A square matrix is said to be Orthogonal if $AA^T = A^T A = I$

- If A is orthogonal, then A^T is also orthogonal.
- If A, B are orthogonal matrices, then AB is orthogonal.
- The eigen values of a orthogonal matrix are of unit modulus.

Unitary Matrix: A square matrix is said to be Unitary matrix if $AA^{\theta} = A^{\theta} A = I$

- If A is a Unitary matrix, then A^T, A^{θ} are also Unitary.
- If A, B are Unitary matrices, then AB is Unitary.
- The determinant of a unitary matrix is of unit modulus.
- The eigen values of a unitary matrix are of unit modulus.

Diagonalisation of the matrix:

If a square matrix A of order n has linearly independent Eigen vectors X_1, X_2, X_3, ----, X_n Corresponding to the n Eigen values λ_1, λ_2, λ_3, ----, λ_n respectively then a matrix P can be found such that $P^{-1}AP$ =D (diagonal matrix)
Define P=($X1, X2, X3, ----, Xn$) then

$$AP=PD \Rightarrow P^{-1}(AP) = (P^{-1}P)D \Rightarrow P^{-1}AP = ID = D$$

Note: Suppose A is a real symmetric matrix with n pair wise distinct eigen values λ_1, λ_2, λ_3, ----, λ_n.Then the corresponding eigen vectors X_1, X_2, X_3, ----, X_n are pair wise orthogonal

Then P = ($e_1, e_2, e_3, \ldots\ldots, e_n$) where $e_1 = \left(\frac{X_1}{\|X_1\|}\right)$ $e_2 = \left(\frac{X_2}{\|X_2\|}\right)$ $e_3 = = \left(\frac{X_3}{\|X_3\|}\right)$

hence P is an orthogonal matrix i.e $P^TP = PP^T = I \Rightarrow P^{-1} = P^T$

$P^{-1}AP$ =D $\Rightarrow$ $P^T A P = D$

Modal and Spectral matrix:

The matrix P in the above result which diagonals the square matrix A is called the modal matrix of A and the resulting diagonal matrix D is known as spectral matrix.

Calculation of powers of Matrix: Let A be the square matrix and
Let P a non-singular matrix such that $D = P^{-1} AP$

$D^2 = P^{-1} A^2 P$

Similarly $D^3 = P^{-1} A^3 P$ Like for n we have $D^n = P^{-1} A^n P$ ------ (1)

To obtain A^n , pre- multiply (1) by P and post - multiply (1) by P^{-1}

Then $$A^n = P\begin{bmatrix} \lambda^n{}_1 & 0 & 0 & --- & 0 \\ 0 & \lambda^n{}_2 & 0 & --- & 0 \\ 0 & 0 & \lambda^n{}_3 & --- & 0 \\ 0 & 0 & 0 & --- & \lambda^n{}_n \end{bmatrix} P^{-1}$$

Quadratic form(Upto Three Variables): A homogeneous polynomial of second degree in n variables $(x_1,x_2,\text{-------},x_n)$ is called a quadratic form. The most general quadratic form is

$$Q = \begin{matrix} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n \\ a_{21}x_1 + a_{21}x_2 + \dots\dots\dots + a_{2n}x_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots\dots + a_{nn}x_n \end{matrix}$$

Where a_{ij} are elements of a field F and we can write $Q=X^TAX = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$

The symmetric matrix is called the matrix of quadratic form

Canonical form:

let $Q=X^TAX$ be a quadratic form in n variables then there exists a real non-singular linear transformation X=PY which transforms $Q=X^TAX$ to the form $d_1 y_1^2 + d_2 y_2^2 + \dots\dots + d_n y_n^2$ this form is called canonical form or normal form

Rank of the quadratic form :

- The number of non zero terms in canonical form.
- It is denoted by r

Index:

- The number of positive square terms in the canonical form.
- It is denoted by s

Signature:

- The difference between the number of positive and negative square terms of be the canonical form.

- Signature=2s-r

Nature of Quadratic form:

- **Positive definite**, if all the Eigen values of A are positive i.e., r = n and s = n.
- **Positive semi definite**, if at least one of the Eigen values of A is zero and other positive i.e., r<n and s = r
- **Negative definite**, if all the Eigen values are negative i.e., r = n and s = 0
- **Negative semi definite**, if at least one of the Eigen values of A is zero and others are negative i.e., r<n and s = 0.
- **Indefinite**, if the Eigen values of A are positive and negative.

Linear transformation: Consider a set of n linear equations

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = y_1$$
$$a_{21}x_1 + a_{21}x_2 + \dots\dots\dots + a_{2n}x_n = y_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{nn}x_n = y_n$$

It can represented as Y=AX
If the above equation is additive and homogenous
The transformation is called linear transformation

Orthogonal Transformation:
Suppose that A is a real square matrix of order n. Then the linear transformation Y=AX is called an orthogonal linear transformation or orthogonal transformation if A is an orthogonal matrix.
Reduction of quadratic form to canonical form:

Method : **Orthogonalization**
Procedure:
Step1: Write the matrix A of the given quadratic form.
Step2: Find the Eigen values λ_1, λ_2 , λ_3 and corresponding Eigen vectors X_1 ,X_2 and X_3 in the normalized form.
Step3: Write the modal matrix P = [X_1 X_2 X_3] formed by normalized vectors.
Step4: P being an orthogonal matrix $P^{-1}=P^T$ so that $P^TAP = D$ where D is the diagonal matrix formed by Eigen values.
Step5: The canonical form is $Y^T (P^TAP) Y = \lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2$
Step6: The orthogonal transformation is X=PY
Note: The matrix A of the quadratic form is symmetric matrix and so diagonalization is by orthogonal transformation.

Example1: Reduce the quadratic form $6x_1^2+3x_2^2+3x_3^2-4x_1x_2+4x_1x_3-2x_2x_3$ to canonical form by an orthogonalization. Find rank, Index and signature of quadratic form.

Sol: The matrix A of the given quadratic form is

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation of the matrix A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 10\lambda^2 + 20\lambda - 32 = 0 \Rightarrow (\lambda-2)(\lambda-2)(\lambda-8) = 0$$

$\Rightarrow \lambda = 2, 2, 8$

The Eigen values of A are 2,2 and 8.

Let the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to the Eigen value λ is given by the non-zero solution of the equation $(A-\lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\lambda=8$, then the Eigen vector X_1 is given by $(A-8I)X = O$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $-2x_1-2x_2+2x_3 = 0$, $-2x_1-5x_2-x_3 = 0$, $2x_1-x_2-5x_3 = 0$

Solving any two of the equations, we get

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} = k \text{ (say)}$$

$\Rightarrow$ $x_1=2k$, $x_2=-k$ and $x_3=k$

For the eigen value λ=8, the eigen vector $X_3 = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix}$

In particular k=1, $X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

If λ=2, the Eigen vector X is given by (A-2I) X=O

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 4x_1-2x_2+2x_3 = 0 \Rightarrow -2x_1+x_2-x_3 = 0 \Rightarrow 2x_1-x_2+x_3 = 0$$

Let $x_2 = k_2$, $x_3 = k_1$, $2x_1 = k_2-k_1$ $\therefore$ $X_2 = \begin{bmatrix} \frac{k_2-k_1}{2} \\ k_2 \\ k_1 \end{bmatrix} = 2\begin{bmatrix} k_2-k_1 \\ 2k_2 \\ 2k_1 \end{bmatrix} = 2k_1\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + 2k_2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\therefore$ Eigen vectors corresponding to λ= 2 are $X_2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\therefore$ Eigen vectors of A are, $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -\alpha+\beta \\ 2\beta \\ 2\alpha \end{bmatrix}$ is also eigen vector of corresponding to λ =2, let us find α, β such that resultrnt vector is orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $(-\alpha+\beta).1+ \quad 2\beta.2+ 2\alpha .0 = 0$

$\Rightarrow \alpha = 5\beta$ Let $\beta = 1$, $\alpha = 5$

$\therefore$ Eigen vectors , $\begin{bmatrix} -4 \\ 2 \\ 10 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ are pair wise orthogonal

[ ][ ]

The normalized Eigen vector of $X_1 = \begin{bmatrix} -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{bmatrix} = e_1$

The normalized Eigen vector of $X_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix} = e_2$

The normalized Eigen vector of $X_3 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} = e_3$

The modal matrix in normalized form P = [e_1 e_2 e_3]

$$= \begin{bmatrix} -\frac{2}{\sqrt{30}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{6}} \\ \frac{5}{\sqrt{30}} & 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$$

=

$$P^T = \begin{bmatrix} \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \therefore P^TAP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \text{diag } [2\ 2\ 8] = D$$

$$Q = Y^TDY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 2y_1^2+2y_2^2+8y_3^2$$

Which is required canonoical form. Quadratic form is reduced to the normal form $2y_1^2+2y_2^2+8y_3^2$ by the orthogonal transformation X = PY. i.e.,

$x_1 = \frac{-2y_1}{\sqrt{30}} + \frac{y_2}{\sqrt{5}} + \frac{2y_3}{\sqrt{6}}$; $x_2 = \frac{y_2}{\sqrt{30}} + \frac{2y_2}{\sqrt{5}} - \frac{y_3}{\sqrt{6}}$; $x_3 = \frac{5y_1}{\sqrt{30}} + \frac{y_3}{\sqrt{6}}$

The rank of quadratic form r = no. of non-zero terms in the canonical form = 3

The Index s = no. of positive terms in the canonical form = 3

Signature = 2s – r = 3.

Nature : positive definite

UNIT-III

Assignment-cum-Tutorial Questions

Section A

Objective Questions

1. The nature of the quadratic form is ____________________if all eigen values of **A** are positive .
2. If the eigen values of A are 0,3,15 then index and signature of X^TAX are ___________
3. If the sum of the eigen values of the matrix of the quadratic form is 0 then nature of the Q.F is _________________________
4. List out the nature of quadratic forms?
5. Discuss the nature of Q.F that is associated with the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
6. Convert the Q.F $x_1^2 - 3x_3^2 + x_1x_2 - 4x_2x_3 + 3x_1x_3$ in matrix form.
7. Symmetric matrix corresponding to the quadratic form $x_1^2 + 6x_1x_2 + 5x_2^2$ _________________

8. Signature of quadratic form is ____________ [ ]
   (a) s (b) r (c) $2s - r$ (d) 2s+r
9. Quadratic form is positive definite if________ [ ]
   (a) r<n and s = 0.(b) r = n and s = n (c) r = n and s = 0 (d) r<n and s = r
10. If the C.F of Q.F is $y_1^2 - 2y_2^2 + y_3^2$ then rank, index and signature of Q.F is [ ]
    (a)1,2,3 (b)3,2,1 (c)3,1,2 (d) 2,3,1
11. If the eigen values of A are -1, 3, 7. Then index and signature of the Q.F X^TAX are [ ]
    (a)2, 1 (b)3,1 (c)3,2 (d)0,3
12. If A is symmetric singular matrix and two of the eigen values are positive then nature of Q.F is [ ]
    (a)+ve definite (b)+ve semi-definite (c)-ve definite (d)in definite
13. I f A= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ then nature of Q.F is [ ]
    (a)+ve definite (b)+ve semi-definite (c)-ve definite (d)in definite
14. If the eigen values of A are 0,0,6 then rank of Q.F Is [ ]
    (a)1 (b)2 (c)3 (d)0

Section B

Descriptive Questions

1. Define Real and Complex matrices.
2. Convert the diagonal matrix diag (m1,m2,------,mn) into quadratic form?
3. Define rank, index, signature of Q.F?
4. Discuss the index and signature of the Q.F $2x^2+2y^2-2z^2$.
5. List out the nature of Q.F?
6. Discuss the linear and orthogonal linear transformations.
7. Write the symmetric matrices corresponding to the following quadratic forms
   i. $ax^2+2hxy+by^2$
   ii. $x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3$
   iii. $x_1^2+2x_2^2+4x_2x_3+x_3x_4$

8. Obtain the quadratic forms corresponding to the following symmetric matrices

   i. $A=\begin{bmatrix}1 & 5\\ 5 & -3\end{bmatrix}$ ii. $B=\begin{bmatrix}2 & 1 & 5\\ 1 & 3 & -2\\ 5 & -2 & 4\end{bmatrix}$ iii. $C=\begin{bmatrix}1 & -2 & 3 & -1\\ -2 & 4 & -6 & 0\\ 3 & -6 & 9 & -3\\ -1 & 0 & -3 & 1\end{bmatrix}$

9. Identify the nature of the quadratic form $6x^2+3y^2+3z^2-4yz+4xz-2xy$.
10. Identify the nature of the quadratic form and find rank, index, signature of the Q.F.
    $x^2+y^2+2z^2-2xy+zx$.
11. Reduce the quadratic form $7x^2+6y^2+5z^2-4xy-4yz$ to the canonical form.
12. Reduce the quadratic form to canonical form by an orthogonal reduction and also find the corresponding linear transformation and its nature ,rank and signature of $2x^2+2y^2+2z^2-2xy+2zx-2yz$.
13. Reduce the quadratic form to sum of squares form by an orthogonal reduction and also find the corresponding linear transformation and its nature, rank and signature of $3x_1^2+3x_2^2+3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3$.
14. Find the orthogonal transformation which transforms the quadratic form $x_1^2+3x_2^2+3x_3^2-2x_2x_3$ to the canonical form.

Learning Material

UNIT-IV: LAPLACE TRANSFORMS

Objectives:

- To know the properties of Laplace transforms
- To know the Transform of one variable function to another variable function.
- To find the Laplace Transform of standard functions

Syllabus: Laplace transforms of standard functions- Properties: Shifting Theorems, change of scale, derivatives, integrals, multiplication and division – Unit step function – Dirac Delta function. Evaluation of improper integrals.

Course Outcomes:

The students is able to

- Calculate the Laplace transform of standard functions both from the definition and by using formulas
- Select and use the appropriate shift theorems in finding Laplace transforms.
- Evaluation of Improper integrals.

Introduction:

Laplace Transform (**LT**) is a powerful technique to replace the operations of calculus by operations of algebra. Integral Transforms play an important role in solving many problems in engineering and real – life problems and several transforms were developed by various mathematicians, statisticians and engineers. Laplace Transform is such an integral transform which is one of the widely applicable transform and is developed by Euler (1737) to solve the differential equations. Later Laplace independently used it in his book on Probability “Theorie Analytique De Probabilities” in ordinary or partial differential equation with boundary values by converting it into simple algebraic equations.

Definition: If $f(t)$ is a function of *exponential order* and $\int_0^\infty e^{-st} f(t)dt$ exists for some $s > 0$ the value of $\int_0^\infty e^{-st} f(t)dt = \bar{f}(s)$ is called the Laplace Transform or one sided Laplace Transform and is denoted by $L\{f(t)\}$.

Symbolically, $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \bar{f}(s)$.

- **Function of exponential order:** If $f(t)$ is defined for all $t > 0$ and there exists constants α and M such that $|f(t)| \le Me^{\alpha t}$ for all t.
- *Note (1):* One sided LTs are unilateral whereas two sided LTs are bilateral Laplace Transforms.
- *Note (2):* A two sided LT obtained by setting the other limit of integral as $-\infty$.

Laplace transforms of some elementary functions:

Let $f(t) = 1$ then $L\{f(t)\} = L(1) = \frac{1}{s}, \quad s > 0$

1. Let $f(t) = e^{at}$ then $L\{f(t)\} = L(e^{at}) = \frac{1}{s-a}, \quad s > a$
2. Let $f(t) = e^{-at}$ then $L\{f(t)\} = L(e^{-at}) = \frac{1}{s+a}, \quad s > -a$.
3. Let $f(t) = t^n$ then $L\{f(t)\} = L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$.
4. Let $f(t) = \sin at$ then $L\{f(t)\} = L(\sin at) = \frac{a}{s^2 + a^2}, \ s > 0.$
5. Let $f(t) = \cos at$ then $L\{f(t)\} = L(\sin at) = \frac{s}{s^2 + a^2}, \ s > 0.$
6. Let $f(t) = \sinh at$ then $L\{f(t)\} = L(\sinh at) = \frac{a}{s^2 - a^2}, \ s > |a|.$
7. Let $f(t) = \cosh at$ then $L\{f(t)\} = L(\sin at) = \frac{s}{s^2 - a^2}, \ s > |a|.$

Properties of Laplace transform:

1. Laplace transform operator L is linear. Laplace transform of a linear combination (sum) of functions is the linear combination (sum) of Laplace transforms of the functions.

2. Change of scale property: When the argument t of f is multiplied by a constant k, s is replaced by s/k in $\bar{f}(s)$ or $F(s)$ and multiplied by $1/k$.
3. First shift theorem proves that multiplication of $f(t)$ by e^{at} amounts to replacement of s by $s-a$ in $\bar{f}(s)$.
4. Laplace transform of a derivative f' amounts to multiplication of $\bar{f}(s)$ by s (approximately but for the constant $-f(0)$).
5. Laplace transform of integral of f amounts to division of $\bar{f}(s)$ by s.
6. Laplace transform of multiplication of $f(t)$ by t^n amounts to differentiation of $\bar{f}(s)$ for n times w.r.t. s (with $(-1)^n$ as sign).
7. Division of $f(t)$ by t amounts to integration of $\bar{f}(s)$ between the limits s to ∞.
8. Second shift theorem proves that the L.T. of shifted function $f(t-a)u(t-a)$ is obtained by multiplying $\bar{f}(s)$ by e^{-at}.

Problems:

1) If f(t) = t^3+4t^2+5, then L[f(t)] = $\frac{\Gamma(4)}{s^4}+4\frac{\Gamma(3)}{s^3}+5\frac{\Gamma(2)}{s^2}=\frac{6}{s^4}+\frac{8}{s^3}+\frac{5}{s^2}$

2) Find Laplace transform of $\sin t\cos 2t$.

Solution: Let $f(t)=\sin t\cos 2t$

$$=\frac{1}{2}(\sin 3t-\sin t)$$

Apply LT on both sides, we have

$L(\sin t\cos 2t)=L\left[\frac{1}{2}(\sin 3t-\sin t)\right]=\frac{1}{2}L(\sin 3t)-\frac{1}{2}L(\sin t)$ (Using linearity property of LT)

$$=\frac{1}{2}\left(\frac{3}{s^2+9}\right)-\frac{1}{2}\left(\frac{1}{s^2+1}\right).$$

3) Find the LT of $e^{-4t}\sin 3t$.

Solution: Let $f(t)=\sin 3t$

By the definition of LT, $L\{\sin 3t\}=\dfrac{3}{s^2+a^2}$

Hence by first shifting theorem, $L\{e^{-4t}\sin 3t\}=\dfrac{3}{(s+4)^2+9}=\dfrac{3}{s^2+8s+25}$.

Laplace transforms of derivatives:

Statement: Let $f(t)$ be a real continuous function which is of exponential order and $f^{'}(t)$ is sectionally continuous and is of exponential order. Then $L\{f^{'}(t)\}=s\bar{f}(s)-f(0)$ Where $\bar{f}(s)=L\{f(t)\}$.

In general,

$L\{f^{(n)}(t)\}=s^n\bar{f}(s)-s^{n-1}f(0)-s^{n-2}f^{'}(0)-s^{n-3}f^{''}(0)-\ldots-f^{(n-1)}(0).$

Laplace transforms of integrals:

Statement: Suppose $f(t)$ is a real function and $g(t)=\int_0^t f(u)du$ is a real function such that both $f(t)$, $g(t)$ satisfy the conditions of existence of Laplace transform then

$L\{g(t)\}=L\left[\int_0^t f(u)du\right]=\dfrac{\bar{f}(s)}{s}$ Where $\bar{f}(s)=L\{f(t)\}$.

Laplace transform of the function $f(t)$ multiplied by t^n:

Statement: If $f(t)$ is sectionally continuous and is of exponential order and if $L\{f(t)\}=\bar{f}(s)$ then $L\{t^n f(t)\}=(-1)^n\dfrac{d^n\bar{f}(s)}{ds^n}$ where $n=1,2,\ldots.$

Laplace transform of the function $f(t)$ divided by t^n:

If $L\{f(t)\}=\bar{f}(s)$ then $L\left(\dfrac{f(t)}{t}\right)=\int_0^\infty \bar{f}(s)ds$ provided $f(t)$ satisfy the condition of existence of LT and the right hand side integral exists.

4) *Problem:* Find the Laplace transform of $f(t) = t\cosh at$, using LT of derivatives.

Solution: We are given $f(t) = t\cosh at$.

It is known that $f^{'}(t) = a\cosh at + at\sinh at$ and

$$f^{''}(t) = 2a\sinh at + a^2 t\cosh at$$

By applying LT on both sides, $L\{f^{''}(t)\} = 2aL\{\sinh at\} + a^2 L\{t\cosh at\}$

By the LT of derivatives, $s^2 L\{f(t)\} - sf(0) - f^{'}(0) = 2a\dfrac{a}{s^2 - a^2} + a^2 L\{t\cosh at\}$

Since $f(0) = 0$ and $f^{'}(0) = 1$, on simplification, we have

$$L\{t\cosh at\} = \frac{2a^2}{(s^2 - a^2)^2}.$$

5) ***Problem:*** Find $L\left(\int_0^t ue^{-u}\sin 4u\,du\right)$.

Solution: Let $f(t) = \sin 4u$

By LT, $L\{\sin 4u\} = \dfrac{4}{s^2 + 4^2} = \dfrac{4}{s^2 + 16}$

By first shifting theorem, $L\{e^{-u}\sin 4u\} = \dfrac{4}{(s+1)^2 + 16} = \dfrac{4}{s^2 + 2s + 17}$

Then by LT of $t^n f(t)$, $L\{ue^{-u}\sin 4u\} = -\dfrac{d}{ds}\left(\dfrac{4}{s^2 + 2s + 17}\right) = \dfrac{4}{(s^2 + 2s + 17)} = \bar{f}(s)$.

Therefore, the LT of integrals, we have

$$L\left(\int_0^t ue^{-u}\sin 4u\,du\right) = \frac{\bar{f}(s)}{s} = \frac{4}{s(s^2 + 2s + 17)}.$$

6) ***Problem:*** Find $L\left(\dfrac{\sin at\cos bt}{t}\right)$.

Solution: Let $f(t) = \sin at\cos bt$

$$= \frac{1}{2}[\sin(a+b)t + \sin(a-b)t]$$

By applying LT on both sides,

$$L\{\sin at\cos bt\} = \frac{1}{2}[L\{\sin(a+b)t\} + L\{\sin(a-b)t\}]$$

$$= \frac{1}{2}.\frac{(a+b)}{s^2+(a+b)^2} + \frac{1}{2}.\frac{(a-b)}{s^2+(a-b)^2} = \bar{f}(s)$$

Now, by the LT of $\frac{f(t)}{t}$, $L\left\{\frac{\sin at\cos bt}{t}\right\} = \frac{1}{2}\int_s^\infty \frac{(a+b)}{k^2+(a+b)^2}ds + \frac{1}{2}\int_s^\infty \frac{(a-b)}{k^2+(a-b)^2}ds$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{k}{a+b}\right)\right]_s^\infty + \frac{1}{2}\left[\tan^{-1}\left(\frac{k}{a-b}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a+b}\right)\right] + \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a-b}\right)\right]$$

$$= \frac{1}{2}\cot^{-1}\left(\frac{s}{a+b}\right) + \frac{1}{2}\cot^{-1}\left(\frac{s}{a-b}\right).$$

Unit Step function:

Definition: Unit step function is defined as $U(t-a) = 0,\ \ t < a$

$=1,\ \ t > a$ i.e. this function jumps by1 at $t = a$.

This function is also known as Heaviside unit function.

Laplace transform of Unit step function $U(t-a)$ is given by

$$L\{U(t-a)\} = \int_0^\infty e^{-st}U(t-a)dt = \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.1dt = \int_a^\infty e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}.$$

Unit impulse function:

Definition: The unit impulse function denoted by $\delta(t-a)$ and is defined by

$$\delta(t-a) = \infty,\ \ t = a$$

$$= 0,\ \ t \neq a$$

So that $\int_0^\infty \delta(t-a)dt = 1 \quad (a \geq 0)$.

If a moving object collide with another object then for a short period of time large force is acting on the other body. To explain such mechanism we make use of unit impulse function, which is also called Dirac Delta function.

Evaluation of improper integrals by Laplace transforms:

7) *Problem:* Evaluate the integral $\int_0^\infty \frac{\cos at - \cos bt}{t} dt$.

Solution: Let $I = \int_0^\infty \frac{\cos at - \cos bt}{t} dt$.

$$= \int_0^\infty \frac{\cos at}{t} dt - \int_0^\infty \frac{\cos bt}{t} dt$$

Clearly the given integral is in the form $\int_0^\infty e^{-st} \frac{f(t)}{t} dt$ with $f_1(t) = \cos at$ and

$f_1(t) = \cos bt$

We observe that $\int_0^\infty e^{-st} \frac{\cos at}{t} dt = \int_s^\infty L(\cos at) ds = \int_s^\infty \frac{s}{s^2 + a^2} ds$ and

$$\int_0^\infty e^{-st} \frac{\cos bt}{t} dt = \int_s^\infty L(\cos bt) ds = \int_s^\infty \frac{s}{s^2 + b^2} ds$$

$$\therefore \int_0^\infty e^{-st} \left(\frac{\cos at - \cos bt}{t} \right) dt = \int_0^\infty \frac{s}{s^2 + a^2} ds - \int_0^\infty \frac{s}{s^2 + b^2} ds = \int_0^\infty \left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right] ds$$

It is clear that the above integral reduces to I when $s = 0$.

Therefore,

$$I = \int_0^\infty \frac{\cos at - \cos bt}{t} dt = \int_0^\infty \left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right] ds = \left[\frac{1}{2} \log(s^2 + a^2) - \frac{1}{2} \log(s^2 + b^2) \right]_0^\infty$$

$$= \frac{1}{2} \left[\log \left(\frac{s^2 + a^2}{s^2 + b^2} \right) \right]_0^\infty = \frac{1}{2} \left[\log 1 - \log \left(\frac{a^2}{b^2} \right) \right] = \frac{1}{2} \log \left(\frac{a^2}{b^2} \right).$$

UNIT-IV

Assignment-cum-Tutorial Questions

Section A

Objective Questions

1. The Laplace transform of $f(t) = \sin^2 2t$ is ________________.
2. If $f(t) = e^{3t}(\sin 2t + \cos 3t)$ then $L\{f(t)\} =$ ________________.
3. If $f(t) = \sin 2t \cos 3t$ then the Laplace transform of $f(t)$ is ________________.
4. If $f(t) = \dfrac{e^{2t} - e^{3t}}{t}$ then $L\{f(t)\} =$ ________________.
5. If $f(t) = t\sin t$ then $L\{f(t)\} =$ ________________.
6. The value of $\int_0^\infty e^{-3t} t\,dt$ is ________________.
7. With usual notations, $L\{e^{at}t^n\} =$ ________________.
8. The Laplace transform of e^{t^3} is ________________.
9. The Laplace transform of $\dfrac{(1-e^t)}{t}$ is ________________.
10. If $L\{f(t)\} = \bar{f}(s) = \dfrac{s}{s^2+1}$, $f(0) = 0$ then $L\{f'(t)\} =$ ________________.
11. If $L\{f(t)\} = \bar{f}(s) = \dfrac{1}{s^2+1}$, $f(0) = 0$ then $L\{f'(t)\} =$ ________________.
12. Suppose $\int_0^3 e^{-st} f(t)dt = 15$ and $f(t+3) = f(t), \forall t$ then $L\{f(t)\} =$ __________.

Section B

Descriptive Questions

1. Find the Laplace transform of f(t)=$\begin{cases} cost, 0 < t < \pi \\ sint, t > \pi \end{cases}$
2. Find the Laplace transform of $\cos 3t.\cos 2t.\cos t$
3. Fin the Laplace transform of Bessel's function $J_0(t) = 1 - \dfrac{t^2}{2^2} + \dfrac{t^4}{2^2 4^2} - \dfrac{t^6}{2^2 4^2 6^2} + ...$
4. Find $L\left(\int_0^t\int_0^t\int_0^t (t\sin t)dtdtdt\right)$
5. Find the LT of $f''(t)$ if $L\{f(t)\} = \dfrac{1}{1+s^2}$, $f(0) = 0$, $f'(0) = 1$.
6. Find the LT of $\frac{sin3t.cost}{t}$
7. Find the LT of $\frac{1-e^{-t}}{t}$
8. Find the LT of t.$e^{2t}.sin3t$
9. Find the LT of t^3.cosat
10. Evaluate $\int_0^\infty t.e^{-t}.sint\,dt$

11.show that $\int_0^\infty e^{-4t}.t^2.sin2t\,dt = \frac{11}{500}$

12.Find the LT of $\int_0^t \frac{e^t.sint}{t}$ dt

SECTION -C

1. The Laplace Transform of cos(ωt) is $\frac{s}{s^2+\omega^2}$ then L(e^{-2t} cos4t) is [ ]

(GATE-2010)

(a) $\frac{s-2}{(s-2)^2+16}$ (b) $\frac{s+2}{(s-2)^2+16}$ (c) $\frac{s-2}{(s+2)^2+16}$ (d) $\frac{s+2}{(s+2)^2+16}$

2. The function f(t) satisfies the differential equation $\frac{d^2f}{dt^2}+f=0$ and the auxiliary conditions, f(0)=0, $\frac{df}{dt}(0)=4$. The Laplace transform of f(t) is given by **(GATE-2009)**

(a) $\frac{2}{s+1}$ (b) $\frac{4}{s+1}$ (c) $\frac{4}{s^2+1}$ (d) $\frac{2}{s^2+1}$

3. The unilateral Laplace transform of f (t) is $\frac{1}{s^2+s+1}$. The unilateral Laplace transform of t f (t) is [ ]

(a) $\frac{-s}{(s^2+s+1)^2}$ (b) $\frac{-2s+1}{(s^2+s+1)^2}$ (c) $\frac{s}{(s^2+s+1)^2}$ (d) $\frac{2s+1}{(s^2+s+1)^2}$

(GATE-2012)

Linear Algebra & Integral transforms

Learning Material

UNIT-V: INVERSE LAPLACE TRANSFORMS

Objectives:

- Understand the properties of Inverse Laplace transforms
- To solve Integral equations by using convolution theorem.
- To convert differential equations into algebraic equations using Laplace Transforms and inverse Laplace transforms.

Syllabus:

Inverse Laplace Transforms – by partial fractions - Convolution theorem(without proof).

Application: Solution of ordinary differential equations.

Subject Outcomes/Unit Outcomes:

After learning this unit, students will be able to:

- Find inverse Laplace Transforms of the functions $\bar{f}(s)$ to obtain $f(t)$.
- Apply convolution theorem to find the Inverse Laplace transform of product of functions.
- Use the method of Laplace transforms to solve systems of linear first-order differential equations.

Definition: Suppose $f(t)$ is a piecewise continuous function and is of exponential order. Let $L\{f(t)\}=\int_0^\infty e^{-st} f(t)dt=\bar{f}(s)$. The inverse Laplace Transform of $f(t)$ is defined as $L^{-1}\{\bar{f}(s)\}=f(t)$, where L^{-1}inverse operator of is L and vice-versa.

Inverse Laplace transforms of some elementary functions:

(1). $L^{-1}\left\{\frac{1}{s}\right\}=1$ (2). $L^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$ (3). $L^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\}=t^n$ (4). $L^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\sin at$

(5). $L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at$ (6). $L^{-1}\left\{\frac{a}{s^2-a^2}\right\}=\sinh at$ (7). $L^{-1}\left\{\frac{s}{s^2-a^2}\right\}=\cosh at$, etc.

Properties of Inverse Laplace transform:

Linear property:

If $L^{-1}\{\bar{f}(s)\}=f(t), L^{-1}\{\bar{g}(s)\}=g(t)$, then $L^{-1}\{a\,\bar{f}(s)+b\,\bar{g}(s)\}=a\,f(t)+b\,g(t)$

Shifting Property:

If $L\{f(t)\}=\bar{f}(s)$ then $L^{-1}\{\bar{f}(s-a)\}=e^{at}f(t),\ s>a$.

Change of scale property:

If $L\{f(t)\}=\bar{f}(s)$ then $L^{-1}\left\{\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)\right\}=f(at)$, then, $L^{-1}\{\bar{f}(as)\}=\frac{1}{a}\bar{f}\left(\frac{t}{a}\right)$.

Problem: let $\bar{f}(s)=\dfrac{4s+4}{4s^2-9}$. Then by linearity property of inverse Laplace transforms (ILT),

$$L^{-1}\left\{\frac{4s+4}{4s^2-9}\right\}=L^{-1}\left\{\frac{4s}{4s^2-9}\right\}+L^{-1}\left\{\frac{4}{4s^2-9}\right\}$$

$$=L^{-1}\left\{\frac{s}{s^2-(3/2)^2}\right\}+L^{-1}\left\{\frac{1}{s^2-(3/2)^2}\right\}=\cosh\frac{3}{2}t+\frac{2}{3}\sinh\frac{3}{2}t$$

Problem: Find the ILT of $\dfrac{4}{(s+1)(s+2)}$.

Solution: Let $\bar{f}(s)=\dfrac{4}{(s+1)(s+2)}$

By applying partial fractions, we can rewrite $\bar{f}(s)$ as

$$\bar{f}(s)=\frac{4}{(s+1)(s+2)}=\frac{A}{(s+1)}+\frac{B}{(s+2)}=\frac{As+2A+Bs+B}{(s+1)(s+2)}$$

Comparing like terms in the numerator, we obtain $A=4$ and $B=-4$.

Therefore, $\bar{f}(s)=\dfrac{4}{(s+1)(s+2)}=\dfrac{4}{(s+1)}-\dfrac{4}{(s+2)}$

By applying linearity property, we have

$L^{-1}\{\bar{f}(s)\}=4L^{-1}\left\{\dfrac{1}{s+1}\right\}-4L^{-1}\left\{\dfrac{1}{s+2}\right\}=4e^{-t}-4e^{-2t}$. (This is the particular solution of an ordinary differential of order 2 whose roots of the auxiliary equation are -1 and -2.)

Problem: Find the ILT of $\dfrac{s+1}{s^2+s+1}$.

Solution: Consider $\bar{f}(s)=\dfrac{s+1}{s^2+s+1}$

$$= \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

By the linearity property of ILT, we have

$$L^{-1}\left(\frac{s+1}{s^2+s+1}\right) = L^{-1}\left(\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right) + L^{-1}\left(\frac{1/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$= e^{-t/2}\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t = e^{-t/2}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right].$$

Inverse Laplace Transforms of Derivatives:

Statement: If $L\{f(t)\} = \bar{f}(s)$ then $L^{-1}\left(\frac{d^n(\bar{f}(s))}{ds^n}\right) = (-1)^n t^n f(t).$

Inverse Laplace Transforms of Integrals:

Statement: If $L\{f(t)\} = \bar{f}(s)$ then $L^{-1}\left(\int_s^\infty \bar{f}(s)ds\right) = \frac{f(t)}{t}.$

Inverse Laplace Transform of type $s\bar{f}(s)$:

Statement: If $L\{f(t)\} = \bar{f}(s)$ and $f(0)=0$ then $L^{-1}\left(s\bar{f}(s)\right) = f'(t)$

Inverse Laplace Transform of type $\frac{\bar{f}(s)}{s}$:

Statement: If $L\{f(t)\} = \bar{f}(s)$ then $L^{-1}\left(\frac{\bar{f}(s)}{s}\right) = \int_0^t f(t)dt$

Similarly, $L^{-1}\left(\frac{\bar{f}(s)}{s^2}\right) = \int_0^t\int_0^t f(t)dt$ and hence in general,

$L^{-1}\left(\frac{\bar{f}(s)}{s^n}\right) = \int_0^t\int_0^t \dots \int_0^t f(t)dt\,dt \dots dt$ (n-folded integral).

Problem: Evaluate $L^{-1}\left\{\frac{s}{\left(s^2+2^2\right)^2}\right\}$?

Solution: We know that $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$, then by derivative property of ILT, we have $L^{-1}\left[\dfrac{-2s}{\left(s^2+a^2\right)^2}\right]=-\dfrac{t}{a}\sin at$, $\therefore L^{-1}\left\{\dfrac{s}{(s^2+2^2)^2}\right\}=\dfrac{t}{4}\sin 2t$.

Convolution Theorem: This is used to find inverse Laplace transforms of product of functions and the operation of convolution between two functions yields another function.

Definition: Suppose $L\{f(t)\}=\bar{f}(s)$ and $L\{g(t)\}=\bar{g}(s)$ then the convolution product of $f(t)$ and $g(t)$ is defined as:

$$f(t)*g(t)=\int_0^t f(r)g(t-r)dr$$, provided the integral exists.

Example: Using convolution theorem find the inverse Laplace transform of $\dfrac{s^2}{(s^2+4)(s^2+9)}$.

Solution: We are given $f(t)=\dfrac{s^2}{(s^2+4)(s^2+9)}$

The given function $f(t)$ can be rewritten as,

$$f(t)=\frac{s^2}{(s^2+4)(s^2+9)}=\frac{s}{(s^2+4)}\cdot\frac{s}{(s^2+9)}$$

By applying inverse Laplace transform, we have,

$$L^{-1}\{f(t)\}=L^{-1}\left\{\frac{s^2}{(s^2+4)}\cdot\frac{s^2}{(s^2+9)}\right\}$$

Hence by convolution theorem,

$L^{-1}\left\{\dfrac{s^2}{(s^2+4)}\cdot\dfrac{s^2}{(s^2+9)}\right\}=(\cos 2t)*(\cos 3t)$ since, $L^{-1}\left(\dfrac{s}{s^2+4}\right)=\cos 2t$ and $L^{-1}\left(\dfrac{s}{s^2+9}\right)=\cos 3t$

$$=\int_0^t[\cos 2u\cos 3(t-u)]du=\int_0^t\frac{1}{2}[\cos(3t-u)+\cos(5u-3t)]du$$

$$=\frac{1}{2}\left[\frac{\sin(3t-u)}{(-1)}\right]_0^t+\frac{1}{2}\left[\frac{\sin(5u-3t)}{5}\right]_0^t=\frac{-1}{2}[\sin 2t-\sin 3t]+\frac{1}{10}[\sin 2t+\sin 3t]$$

$$=\sin 2t\left(-\frac{1}{2}+\frac{1}{10}\right)+\sin 3t\left(\frac{1}{2}+\frac{1}{10}\right)=\frac{1}{5}(3\sin 3t-2\sin 2t).$$

Solution of Ordinary differential equation (An application):

Problem: Solve the differential equation $\dfrac{d^3y}{dt^3}+2\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}-2y=0;$ given $y(0)=y'(0)=0$ and $y''(0)=6$.

Solution: We are given the linear non-homogeneous differential equation with constant coefficients:

$$\frac{d^3y}{dt^3}+2\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=0 \text{ where } y=y(t) \text{ or } f(t)$$

Applying Laplace transform on both sides,

$$L\left(\frac{d^3y}{dt^3}\right)+2L\left(\frac{d^2y}{dt^2}\right)-L\left(\frac{dy}{dt}\right)-2L(y)=L(0)$$

$$\Rightarrow \left[s^3\bar{f}(s)-s^2f(0)-sy'(0)-y''(0)\right]+2\left[s^2\bar{f}(s)-sy(0)-y'(0)\right]-\left[s\bar{f}(s)-y(0)\right]-2\bar{f}(s)=0$$

$$\Rightarrow \bar{f}(s)\left[s^3+2s^2-s-2\right]-y(0)\left[s^2+2s-1\right]-y'(0)(s+2)-y''(0)=0$$

Substituting $y(0)=y'(0)=0$ and $y''(0)=6$, we get,

$$\bar{f}(s)(s^3+2s^2-s-2)-6=0$$

$$\Rightarrow \bar{f}(s)=\frac{6}{(s^3+2s^2-s-2)}$$

Now by applying inverse Laplace transform on both sides,

$$L^{-1}(\bar{f}(s))=L^{-1}\left(\frac{6}{s^3+2s^2-s-2}\right)=L^{-1}\left(\frac{6}{s^2(s+2)-(s+2)}\right)$$

$$f(t)=L^{-1}\left(\frac{6}{(s+2)(s+1)(s-1)}\right)$$

Consider $\bar{f}(s)=\dfrac{6}{(s-1)(s+1)(s+2)}=\dfrac{A}{(s-1)}+\dfrac{B}{(s+1)}+\dfrac{C}{(s+2)}$

On simplification we obtain $A=1,\quad B=-3,\quad C=2$

$$\therefore \quad L^{-1}(\bar{f}(s))=f(t)=L^{-1}\left(\frac{1}{s-1}\right)-L^{-1}\left(\frac{3}{s+1}\right)+L^{-1}\left(\frac{2}{s+2}\right)$$

$$=e^t-3e^{-t}+2e^{-2t}$$

Hence, the solution of the given differential equation is $y(t)=e^t-3e^{-t}+2e^{-2t}$.

Problem: Solve the differential equation $t\dfrac{d^2y}{dt^2}+(1-2t)\dfrac{dy}{dt}-2y=0$ where $y(0)=1,\ y'(0)=2$.

Solution: We are given the linear differential equation with variable coefficients:

$$t\frac{d^2y}{dt^2}+(1-2t)\frac{dy}{dt}-2y=0$$

Applying Laplace transform on both sides,

$$L\left(t\frac{d^2y}{dt^2}\right)+L\left((1-2t)\frac{dy}{dt}\right)-2L(y)=0$$

$$\Rightarrow -\frac{d}{ds}\left(s^2\bar{f}(s)-sf(0)-f^{'}(0)\right)+\left(s\bar{f}(s)-f(0)\right)+2\frac{d}{ds}\left(s\bar{f}(s)-f(0)\right)-2\bar{f}(s)=0$$

$$\Rightarrow \bar{f}^{'}(s)(2s-s^2)-s\bar{f}(s)=0$$

$$\Rightarrow \frac{\bar{f}^{'}(s)}{\bar{f}(s)}=-\frac{1}{s-2}$$

Integrating on both sides, we have,

$$\log\bar{f}(s)=-\log(s-2)+\log c$$

$$\Rightarrow \bar{f}(s)=\frac{c}{s-2}$$

By applying inverse Laplace transform on both sides,

$$L^{-1}(\bar{f}(s))=L^{-1}\left(\frac{c}{s-2}\right)$$

$$\Rightarrow f(t)=ce^{2t}$$

By using the initial condition, we have $c=1$.

Therefore, the particular solution of the differential equation is $f(t)=e^{2t}$.

UNIT-V

Assignment-cum-Tutorial Questions

Section A

Objective Questions

1. $L^{-1}\left(\frac{1}{s}\right) =$ [ ]

a) 0 b) 1 c) t d) 1 / t

2. $L^{-1}\left(\frac{1}{s^2+a^2}\right) =$ [ ]

(a) $\sin at$ (b) $\cos at$ (c) $\frac{1}{a}\sin at$ (d) $\frac{1}{a}\cos at$

3. $L^{-1}\left(\frac{1}{3s-6}\right) =$ [ ]

(a) e^{6t} (b) $\frac{1}{3}e^{2t}$ (c) e^{2t} (d) does not exist

4. $L^{-1}\left(\frac{1}{(s+a)(s+b)}\right) =$ [ ]

(a) $\frac{e^{at}-e^{bt}}{b-a}$ (b) $\frac{e^{-at}+e^{-bt}}{b-a}$ (c) $\frac{e^{-at}-e^{-bt}}{b-a}$ (d) $\frac{e^{at}+e^{bt}}{b-a}$

5. $L^{-1}\left(\frac{s+2}{(s-2)^2}\right) =$ [ ]

(a) $e^{2t}(1+2t)$ (b) $te^{2t}(1+2t)$ (c) $(1+2t)$ (d) $t(1+2t)$

6. $L^{-1}\left(\frac{s+2}{s^2-2s+5}\right) =$ [ ]

(a) $\cos 2t+\frac{3}{2}\sin 2t$ (b) $\sin 2t+\frac{3}{2}\cos 2t$ (c) $e^t\cos 2t+\frac{3}{2}e^t\sin 2t$ (d) $\cos 2t$

7. $L^{-1}\left(\int_s^\infty \bar{f}(s)ds\right) =$ [ ]

(a) $\frac{f(t)}{t}$ (b) $\int_0^t f(t)dt$ (c) $\int_0^t \frac{f(t)}{t}dt$ (d) $f(t)$

8. $L^{-1}\left\{\frac{1}{s^4}\right\} =$ [ ]

a) $\frac{t^3}{3}$ b) $\frac{t^3}{6}$ c) $\frac{t^4}{3}$ d) $\frac{t^4}{6}$

9. $L^{-1}\left\{\frac{2s-5}{s^2-9}\right\} =$ [ ]

a) 2cosh3t – 5sinh3t b) $2\cos 3t + \frac{5}{3}\sin 3t$

c) $2\cosh 3t + \frac{5}{3}\sinh 3t$ d) $2\cosh 3t - \frac{5}{3}\sinh 3t$

10. $L^{-1}\left\{\frac{1}{(s-a)^3}\right\} =$ [ ]

a) $\frac{t^2}{2}e^{at}$ b) $\frac{t}{2}e^{at}$ c) $\frac{t}{2}e^{-at}$ d) $\frac{t^2}{2}e^{-at}$

11. $L^{-1}\left\{\frac{e^{-3s}}{s^3}\right\} =$ [ ]

a) $\frac{(t-3)\mu(t-3)^2}{2}$ b) $\frac{(t-3)\mu(t-3)}{2}$ c) $\frac{(t-3)^2\mu(t-3)}{2}$ d) $(t-3)\mu(t-3)$

12. $L^{-1}\left\{\frac{s^2+3s+7}{s^3}\right\} =$ [ ]

a) $1+3t+\frac{7}{2}t^2$ b) $1+\frac{3}{2}t^2+\frac{7}{3}t^2$ c) $1+3t+\frac{7}{3}t^2$ d) $1+\frac{3}{2}t+\frac{7}{3}t^2$

13. $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} =$ [ ]

a) $\frac{t}{a}$ sinat b) $\frac{t}{2a}$ cosat c) $\frac{t}{2a}$ sinat d) $\frac{t}{a}$ cosat

14. $L^{-1}\left\{\frac{d^n}{ds^n}\bar{f}(s)\right\} =$ [ ]

a) $t^n f(t)$ b) $(-1)^n t^n f(t)$ c) $-t^n f(t)$ d) $\frac{(-1)^n t^n}{n!} f(t)$

15. If $(D^2 + 1)x = t$ where $x = 0 = \frac{dx}{dt}$ at $t = 0$ then $L\{x\} =$ [ ]

a) $\frac{1}{s(s^2+1)}$ b) $\frac{1}{s^2(s^2-1)}$ c) $\frac{1}{s^2(s^2+1)}$ d) $\frac{s^2}{s^2+1}$

16. If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $f(0) = 0$ then $L^{-1}\{s\bar{f}(s)\} =$

a) $f^1(t)$ b) $-f^1(t)$ c) $f^1(0)$ d) $-f^1(0)$

Section B

Subjective Questions

1. Find the inverse Laplace theorem of $\frac{1}{s(s+a)(s+b)}$.
2. Solve the differential equation $(D^2+2D+5)y = e^t \sin t;\ y(0)=0,\ y'(0)=1$.
3. Find $L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$
4. Find the inverse Laplace transform of $\frac{1}{(s^2+1)(s-1)(s+5)}$
5. Evaluate $L^{-1}\left\{\frac{1}{s^2(s+2)}\right\}$
6. Find the inverse Laplace Transform of $\frac{s^2}{(s^2+4)(s^2+25)}$
7. Find the inverse Laplace Transform of $\frac{5s-2}{s^2(s+2)(s-1)}$
8. Find $L^{-1}\left[\frac{s}{(s^2+1)((s^2+9)(s^2+25)}\right]$
9. Apply Convolution theorem to evaluate $L^{-1}\{\frac{1}{(S-2)(S+2)^2}\}$
10. Using Laplace transform solve $(D^2+2D+1)x = 3te^{-t}$, given that $x(0)=4$, $\frac{dx}{dt}=0$ at $t=0$
11. Using Laplace Transform solve $(D^2+1)x = t\cos 2t$ given $x=0$, $\frac{dx}{dt}=0$ at $t=0$
12. Using Laplace Transform solve $\frac{d^2y}{dt^2}+2\frac{dy}{dx}-3y=\sin t$, $y=\frac{dy}{dt}=0$, when $t=o$
13. Using Laplace transforms, solve the differential equation $\frac{d^2x}{dt^2}+9x=\sin t$ given $X(0)=1$, $x^1(0)=0$

<u>Sections : C</u>

1. The function f(t) satisfies the differential equation $\frac{d^2f}{dt^2}+f=0$ and the auxiliary conditions, $f(0)=0$, $\frac{df}{dt}(0)=4$. The Laplace transform of f(t) is given by

(GATE-2009)

(a) $\frac{2}{s+1}$ (b) $\frac{4}{s+1}$ (c) $\frac{4}{s^2+1}$ (d) $\frac{2}{s^2+1}$

2. The inverse Laplace transform of the function $F(s)=\frac{1}{s(s+1)}$ is given by **(GATE-2007)**

(a) f(t)=sint (b) f(t) = e^{-t} sint (c) e^{-t} (d) 1- e^{-t}

3. The inverse Laplace transform of F(s) = s+1/(s^2+4) is **(GATE-2011)**

(a) cos2t+ sin2t (b) cos2t-(1/2) sin2t
(c) cos2t+(1/2)sin2t (d) cos2t-sin2t

Unit – VI
FOURIER TRANSFORMS

Objectives:

To introduce

- Fourier transform of a given function and the corresponding inverse.
- Fourier sine and cosine transform of a given function and their corresponding inverses.
- Finite Fourier transforms of a given function and their corresponding inverses.

Syllabus:

Fourier integral theorem (only statement) – Fourier transform – sine and cosine transforms – properties – inverse Fourier transforms.

Outcomes:

Students will be able to

- Find the Fourier transform of the given function in infinite cases.
- Find the Fourier sine and cosine transforms of the given function in infinite cases.

Learning Material

Fourier Transforms are widely used to solve Partial Differential Equations and in various boundary value problems of Engineering such as Vibration of Strings, Conduction of heat, Oscillation of an elastic beam, Transmission lines etc.

Integral Transforms:

- The Integral transform of a function f(x) is defined as

$$I\{f(x)\} = \bar{f}(s) = \int_{x=x_1}^{x_2} f(x)K(s,x)\,dx$$

Where K(s,x) is a known function of s & x, called the 'Kernel' of the transform.

The function f(x) is called the Inverse transform of $\bar{f}(s)$

1. **Laplace Transform**: When K(s,x) = e^{-sx}

$$L\{f(x)\} = \bar{f}(s) = \int_0^{\infty} f(x)e^{-sx}\,dx$$

2. **Fourier Transform**: When K(s,x) = e^{isx}

F{f(x)} = $\bar{f}(s) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)e^{isx}dx$

3. **Fourier Sine Transform**: When K(s,x)=Sinsx

F_s\{f(x)\} = $\bar{f}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)sinsx\ dx$

4. **Fourier Cosine Transform**: When K(s,x)=CosSx

F_c\{f(x)\} = $\bar{f}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)cossx\ dx$

Fourier Integral Theorem:- If f(x) satisfies Dirichlet's conditions for expansion of Fourier series in (-c,c) and $\int_{-\infty}^{\infty}|f(x)|$ converges, then

$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)cos\lambda(t-x)dt\,d\lambda$

is known as Fourier Integral of f(x)

Fourier Sine & Cosine Integrals:-

If f(x) satisfies Dirichlet's conditions for expansion of Fourier series in (-c,c) and $\int_{-\infty}^{\infty}|f(x)|$ converges,

- If f(t) is odd function then f(x) = $\frac{2}{\pi}\int_0^\infty sin\lambda x\ \int_0^\infty f(t)sin\lambda t\ dt\ d\lambda$

is called "Fourier sine Integral" .

- if f(t) is even function then $f(x) = \frac{2}{\pi}\int_0^\infty cos\lambda x\ \int_0^\infty f(t)cos\lambda t\ dt\ d\lambda$

This is called "Fourier cosine Integral"

Complex form of Fourier Integral:-

- The complex form of Fourier integral is known as

$= \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)\ e^{i\lambda(t-x)}dt\ d\lambda$

Problems:

1. Using Fourier integral show that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}d\lambda, a,b > 0$$

Solution: since the integrand on R.H.S contains sine term, we use Fourier sine integral formula.

We know that fouries sine integral for f(x) is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin px \int_0^\infty f(t)\sin pt dt dp$$

Replacing p with λ we get

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\int_0^\infty f(t)\sin\lambda t\,dt\,d\lambda$$

Here f(x) = e^{-ax}-e^{-bx}

f(t) = e^{-at}-e^{-bt}

substituting (2) in (1) , we get

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left(\int_0^\infty \left(e^{-at}-e^{-bt}\right)\sin\lambda t\,dt\right)d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\left\{\frac{e^{-at}}{\lambda^2+a^2}(-a\sin\lambda t-\lambda\cos\lambda t)\right\}_0^\infty-\left\{\frac{e^{-bt}}{\lambda^2+b^2}(-b\sin\lambda t-\lambda\cos\lambda t)\right\}_0^\infty\right]d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda}{\lambda^2+a^2}-\frac{\lambda}{\lambda^2+b^2}\right]d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda\left(b^2-a^2\right)}{\left(\lambda^2+a^2\right)\left(\lambda^2+b^2\right)}\right]d\lambda$$

$$e^{-ax}-e^{-bx}=\frac{2\left(b^2-a^2\right)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{\left(\lambda^2+a^2\right)\left(\lambda^2+b^2\right)}d\lambda$$

Fourier Transforms:-

- The fourier transform of a function f(x) is given by F(s) = $\int_{-\infty}^{\infty} f(x)e^{isx}dx$
- The inverse fourier transform of F(S) is given by f(x) = $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$

Fourier Sine transforms:-

- The Fourier sine transform of f(x) is defined as

  $F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$
- The inverse Fourier sine transform of $F_S(S)$ is defined as f(x) = $\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds$

here $F_s(s)$ is called **Fourier sine transform** of **f(x)** and **f(x)** is called **Inverse Fourier sine transform** of $F_s(s)$

Fourier Cosine transforms

- The Fourier cosine transform of f(x) is defined as

  $F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

- The inverse Fourier cosine transform of $F_S(S)$ is defined as f(x) = $\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\, ds$

here $F_c(s)$ is called **Fourier cosine transform** of **f(x)** and **f(x)** is called **Inverse Fourier cosine transform** of $F_c(s)$

NOTE: 1. Some authors define F.T as follows

i) F(s) = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t)e^{-ist}dt$ ii) f(x) = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(s)e^{-isx}ds$

iii)F(s) = $\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty f(x)e^{-isx}dx$ iv) f(x) = $\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty F(s)e^{isx}ds$

2.Some authors define Fourier sine & cosine transforms as follows

i) $F_s(s) = \int_0^\infty f(x)\sin sx\, dx$ ii) f(x) = $\frac{2}{\pi}\int_0^\infty F_s(s)\sin sx\, ds$

iii) $F_c(s) = \int_0^\infty f(x)\cos sx\, dx$ iv) f(x) = $\frac{2}{\pi}\int_0^\infty F_c(s)\cos sx\, ds$

Properties of Fourier Transforms:-

1. **Linearity Property**:- If $F_1(s)$ and $F_2(s)$ be the Fourier transforms of $f_1(x)$ and $f_2(x)$
2. respectively then f{a$f_1(x) + b\, f_2(x)\} = a\, F_1(s) + b\, F_2(s)$, where a & b are constants
3. .**Change of Scale Property**:- If F{f(x)} = F(s) then F{f(ax)} = $\frac{1}{a}F(\frac{s}{a})$
4. **Shifting Property**:- If F{f(x)} = F(s) then F{f(x-a)} = $e^{isa}F(s)$
5. **Modulation Property**:- If F{f(x)} = F(s) then F{f(x)cosax} = ½ {F(s+a)+F(s-a)}
6. If F{f(x)} = F(s) then F{f(-x)}= F(-s)
7. $\overline{F\{f(x)\}} = F(-s)$
8. $\overline{F\{f(-x)\}} = \overline{F(s)}$
9. $F_c\{xf(x)\} = \frac{d}{ds}F_s\{f(x)\}$

Problem : Derive the relation between Fourier transform and Laplace transform**.**

Solution: consider $f(t) = \begin{cases} e^{-xt}g(t), t>0,, \\ 0, t<0 \end{cases}$

The fourier trasform of f(x) is given by

$$F(f(t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-xt} g(t) e^{ist} dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{(is-x)t} g(t) dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-pt} g(t) dt \quad \textbf{where p=x-is}$$

$$=\frac{1}{\sqrt{2\pi}} L(g(t)) \because \left[L\left(f(t)=\int_{0}^{\infty} e^{-st} f(t) dt \right) \right]$$

$\because$ **Fourier transform of** $f(t)=\frac{1}{\sqrt{2\pi}}\times$ **laplace transform of g(t)**

Problems:

Find the F.T of f(x) = $e^{-|x|}$

sol: Given f(x) = $e^{-|x|}$

$$=\begin{cases} e^{x}; x<0 \\ e^{-x}; x>0 \end{cases}$$

by definition, F{f(x)} = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$=\frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{0} f(x) e^{isx} dx+\int_{0}^{\infty} f(x) e^{isx} dx\right\}$$

$$=\frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{0} e^{(1+is)x} dx+\int_{0}^{\infty} e^{(-1+is)x} dx\right\}$$

$$=\frac{1}{\sqrt{2\pi}}\left\{\left(\frac{e^{(1+is)x}}{1+is}\right)_{-\infty}^{0}+\left(\frac{-e^{-(1-is)x}}{1-is}\right)_{0}^{\infty}\right\}$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+is}+\frac{1}{1-is}\right)$$

$$=\sqrt{\frac{2}{\pi}}\cdot\frac{1}{1+s^2}$$

Problem: Find the Fourier transform of f(x) defined by $f(x)=\begin{cases} 1, \ if\ |x|<a \\ 0, \ if\ |x|>a \end{cases}$

And hence evaluate $\int_{0}^{\infty}\frac{\sin p}{p} dp$ and $\int_{-\infty}^{\infty}\frac{\sin ap\cos px}{p} dp$

Sol: We have F[f(x)]= $\int_{-\infty}^{\infty} e^{ipx} f(x) dx=\int_{-\infty}^{-a} e^{ipx} f(x) dx+\int_{-a}^{a} e^{ipx} f(x) dx+\int_{a}^{\infty} e^{ipx} f(x) dx$

$$=\int_{-a}^{a} e^{ipx} dx=\frac{2\sin ap}{p}$$

By the inversion formula, we know that f(x)= $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx} F(p) dp$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}\frac{2\sin ap}{p} dp = \begin{cases} 1, \ if\ |x|<a \\ 0, \ if\ |x|>a \end{cases}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos px\frac{2\sin ap}{p}dp - \frac{1}{2\pi}\int_{-\infty}^{\infty}\sin px\frac{2\sin ap}{p}dp = \begin{cases}1, & if\ |x|<a\\ 0, & if\ |x|>a\end{cases}$$

Since the second integral is an odd function, $\int_{-\infty}^{\infty}\frac{\sin ap\cos px}{p}dp = \quad \pi$,

$$\begin{cases}1, & if\ |x|<a\\ 0, & if\ |x|>a\end{cases}$$

Put x=0, we get, $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin ap}{p}dp = \begin{cases}1, & if\ \ a>0\\ 0, & if\ \ a<0\end{cases}$

$$\int_0^{\infty}\frac{\sin p}{p}dp = \frac{\pi}{2}, \quad a>0$$

$$= 0, \quad a<0$$

And put x=0 and a=1 then we get $\int_0^{\infty}\frac{\sin p}{p}dp = \frac{\pi}{2}$

Problem: Find the fourier sine transform of e^{-ax}, a>0 and hence deduce that $\int_0^{\infty}\frac{p\sin px}{a^2+p^2}dp$

Sol: $F_s\{f(x)\} = \int_0^{\infty}f(x)\sin px\,dx = \int_0^{\infty}e^{-ax}\sin px\,dx = \frac{p}{a^2+p^2}$

By the inversion formula, we know that f(x)= $\frac{2}{\pi}\int_0^{\infty}F_s\{f(x)\}\sin px\,dp$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{p}{a^2+p^2}\sin px\,dp$$

$$\therefore \int_0^{\infty}\frac{p\sin px}{a^2+p^2}dp = \frac{\pi}{2}e^{-ax}$$

Problem : Find the Fourier sine Transform of $\frac{1}{x}$.

Sol: $F_s\{f(x)\} = \int_0^{\infty}f(x)\sin px\,dx = \int_0^{\infty}\frac{1}{x}\sin px\,dx$

Let px=θ

Pdx=dθ , θ:0$\rightarrow\infty$

$$F_s\{f(x)\} = \int_0^{\infty}\frac{p}{\theta}\sin\theta\frac{d\theta}{p}$$

$$= \sqrt{\frac{\pi}{2}}$$

Problem : Find the Fourier cosine transform of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Solution : $F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx = \int_0^\infty \cos x \cos px\,dx$

$$= \int_0^\infty \frac{\cos(p+1)x + \cos(p-1)x}{2}\,dx$$

$$= \frac{\sin(p+1)a}{p+1} + \frac{\sin(p-1)a}{p-1}$$

Unit – VI

Assignment-Cum-Tutorial Questions

SECTION-A

Objective Questions

1. The complex form of Fourier integral of f(x) is ________________.
2. Fourier Integral of f(x) is __________________________.
3. Fourier transform of f(x) is _________________________.
4. The inverse Fourier transform of $F(s)$ is

   (a) $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{isx}\, ds$ (b) $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, dx$

   (c) $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$ (d) None.

5. Fourier Sine integral of f(x) is _______________________.
6. Fourier Sine Transform of f(x) is _____________________.
7. The inverse Fourier sine transform of $f(x)$ is

   (a) $\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)\cos sx\, ds$ (b) $\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)\cos sx\, dx$

   (c) $\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)\sin sx\, ds$ (d) None.

8. Fourier Cosine integral of f(x) is _____________________.
9. Fourier Cosine transform of f(x) is ___________________.
10. The inverse Fourier cosine transform of $f(x)$ is

    (a) $\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)\cos sx\, ds$ (b) $\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^{\infty} F_c(s)\cos sx\, dx$

    (c) $\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)\cos s\, dx$ (d) None.

11. Finite Fourier Sine transform of f(x) is ________________.
12. The inverse finite Fourier sine transform of $F_s(n)$ is

    (a) $\sum F_s(n)\sin\frac{n\pi x}{c}$ (b) $\frac{2}{c}\sum F_s(n)\sin\frac{n\pi x}{c}$

    (c) a and b (d) None.

13. Finite Fourier Cosine transform of f(x) is _____________**.**

SECTION-B

Subjective Questions

1. Find the Fourier transform of $f(x)=\begin{cases} e^{ikx} & a<x<b \\ 0 & x<a,\ x>b \end{cases}$.
2. Find the Fourier transform of $f(x)=\begin{cases} 1, & |x|<a \\ 0, & |x|>a \end{cases}$, hence evaluate $\int_0^{\infty}\frac{S\,\text{int}}{t}\,dt$.

3. Find the Fourier transform of $f(x)=e^{-x^2/2}, -\infty<x<\infty$ [or] S.T Fourier transform of $e^{-x^2/2}$ is self reciprocal.

4. Find the Fourier transform of f(x) defined by $f(x)=\begin{cases} a^2-x^2, if\ |x|\le 1 \\ 0, \quad if\ |x|>1 \end{cases}$.

And S.T. $\int_0^\infty \frac{\sin t - t\cos t}{t^3}dt=\frac{\pi}{4}$

5. Find the Fourier cosine and sine transform of $5e^{-2x}+2e^{-5x}$

6. Find the a) Fourier cosine and b) Fourier Sine transform of f(x) = e^{-ax} for x≥0 and a>0. And hence deduce the integrals known as "Laplace integrals" $\int_0^\infty \frac{\cos \alpha x}{\alpha^2+a^2}d\alpha$ and $\int_0^\infty \frac{\alpha.Sin\,\alpha x}{\alpha^2+a^2}d\alpha$

7. Find the inverse Fourier cosine transform f(x) if

$$F_c(\alpha)=\begin{cases} \frac{1}{2a}(a-\frac{\alpha}{2}), & \alpha<2a \\ 0, & \alpha\ge 2a \end{cases}$$

8. Find Fourier sine transform f(x)=$e^{-|x|}$ & hence find $\int_0^\infty \frac{x.\sin mx}{1+x^2}dx$

9. Find the Fourier cosine and sine transform of xe^{-ax}.

10. Find the Fourier sine and cosine transforms of $f(x)=\frac{e^{-ax}}{x}$ S.T.

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}Sinsxdx=\tan^{-1}\left(\frac{s}{a}\right)-\tan^{-1}\left(\frac{s}{b}\right).$$

11. Using Fourier integral, Show that $e^{-ax}-e^{-bx}=\frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}d\lambda$

12. Using Fourier integral, prove that $\int_0^\infty \frac{(\alpha^2+2)\cos\alpha x}{\alpha^2+4}d\alpha=\frac{\pi}{2}e^{-x}\cos x$

Section-C

1. The value of the integral [GATE **2014]**

$$\int_{-\infty}^{\infty} \sin c^2(dt) \quad is\ \_\_\_\_\_\_\_.$$

2. . Let $g(t)=e^{-\pi t^2}$, and $h(t)$ is filter marched to $g(t)$. If $g(t)$ is applied as input to $h(t)$, then the Fourier transform of the output is

(a) $e^{-\pi t^2}$ (c) $e^{-\pi|f|}$

(b) $e^{-\pi f^2/2}$ (d) $e^{-2\pi f^2}$

[GATE2013]

3. .The Fourier transform of a signal h(t) is $H(j\omega)=(2\cos\omega)(\sin 2\omega)/\omega$. The value of h(0) is

(a) 1/4 (c) 1

(b) 1/2 (d) 2

[GATE2012]

4.

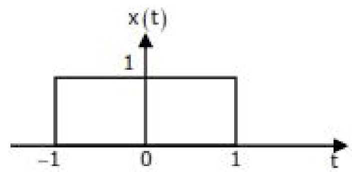
$x(t)$ is a positive rectangular pulse from t = -1 to t = +1 with unit height as shown in the figure. The value of $\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$ {where $X(\omega)$ is the Fourier transform of $x(t)$} is

(A) 2

(B) 2π

(C) 4

(D) 4π



[GATE2010]