GUDLAVALLERU ENGINEERING COLLEGE

(An Autonomous Institute with Permanent Affiliation to JNTUK, Kakinada)

Seshadri Rao Knowledge Village, Gudlavalleru – 521 356.

Department of Computer Science and Engineering

HANDOUT

on

NM & DE

Vision :

To be a Centre of Excellence in computer science and engineering education and training to meet the challenging needs of the industry and society

Mission:

- To impart quality education through well-designed curriculum in tune with the growing software needs of the industry.
- To be a Centre of Excellence in computer science and engineering education and training to meet the challenging needs of the industry and society.
- To serve our students by inculcating in them problem solving, leadership, teamwork skills and the value of commitment to quality, ethical behavior & respect for others.
- To foster industry-academia relationship for mutual benefit and growth

Program Educational Objectives :

PEO1 : Identify, analyze, formulate and solve Computer Science and Engineering problems both independently and in a team environment by using the appropriate modern tools.

PEO2 : Manage software projects with significant technical, legal, ethical, social, environmental and economic considerations.

PEO3 : Demonstrate commitment and progress in lifelong learning, professional development, leadership and Communicate effectively with professional clients and the public

HANDOUT ON NM & DE

Class & Semester:I B.Tech – II Semester Year:2019-20

Branch : CSE Credits : 4

1. Brief history and current developments in the subject area

"MATHEMATICS IS THE MOTHER OF ALL SCIENCES", It is a necessary avenue to scientific knowledge , which opens new vistas of mental activity. A sound knowledge of engineering mathematics is essential for the Modern Engineering student to reach new heights in life. So students need appropriate concepts, which will drive them in attaining goals.

Importance of mathematics in engineering study :

Mathematics has become more and more important to engineering Science and it is easy to conjecture that this trend will also continue in the future. In fact solving the problems in modern Engineering and Experimental work has become complicated, time – consuming and expensive. Here mathematics offers aid in planning construction, in evaluating experimental data and in reducing the work and cost of finding solutions.

2. Pre-requisites, if any

- Basic Knowledge of Mathematics at Intermediate Level is required.

3. Course objectives:

- To understand the various numerical techniques.
- To aware of different techniques to solve first and second order differential equations.

4. Course outcomes:

Upon successful completion of the course, the sstudents will be able to

CO1: apply numerical techniques for solutions of Algebraic, transcendental and ordinary differential equations.

CO2: find interpolating polynomial for the given data.

CO3: apply the learnt techniques to solve first and second order differential equations in various engineering problems.

CO4: find the maximum and/or minimum points on a given surface.

5. Program Outcomes:

Graduates of the Computer Science and Engineering Program will have

a) an ability to apply knowledge of mathematics, science, and engineering

b) an ability to design and conduct experiments, as well as to analyze and interpret data

c) an ability to design a system, component, or process to meet desired needs within realistic constraints such as economic, environmental, social, political, ethical, health and safety, manufacturability, and sustainability

d) an ability to function on multidisciplinary teams

e) an ability to identify, formulate, and solve engineering problems

f) an understanding of professional and ethical responsibility

g) an ability to communicate effectively

h) the broad education necessary to understand the impact of engineering solutions in a global, economic, environmental, and societal context

i) a recognition of the need for, and an ability to engage in life-long learning,

j) a knowledge of contemporary issues

k) an ability to use the techniques, skills, and modern engineering tools necessary for engineering practice.

6. Mapping of Course Outcomes with Program Outcomes:

	a	b	c	d	e	f	G	h	I	j	K
CO1	M				M						
CO2	M				M						
CO3	M				M						
CO4	M				M						

7. Prescribed Text books

- B.S.Grewal, Higher Engineering Mathematics : 42nd edition, Khanna Publishers,2012 , New Delhi.
- B.V.Ramana, Higher Engineering Mathematics, Tata-Mc Graw Hill company Ltd..

8. Reference books

1. U.M.Swamy, A Text Book of Engineering Mathematics – I & II : 2nd Edition, Excel Books, 2011, New Delhi.
2. Dr. T.K.V.Iyengar, Dr. B.Krishna Gandhi, S.Ranganatham and Dr.M.V.S.S.N.Prasad, Engineering Mathematics, Volume-I: 11th edition, S.Chand Publishers, 2012, New Delhi.
3. Erwin Kreyszig, Advanced Engineering Mathematics: 8th edition, Maitrey Printech Pvt. Ltd, 2009, Noida.
4. S. Armugam, A. Thangapandi Isac, A. Soma Sundaram, Numerical Methods, Scitech Publications.

9. Lecture Schedule / Lesson Plan

<table>
<tr><td rowspan="2">Topic</td><td colspan="2">No. of Periods</td></tr>
<tr><td>Theory</td><td>Tutorial</td></tr>
<tr><td colspan="3">UNIT-I:SolutionofAlgebraicandTranscendentalEquations</td></tr>
<tr><td>Introduction</td><td>1</td><td rowspan="3">1</td></tr>
<tr><td>Bisection Method</td><td>2</td></tr>
<tr><td>Method of False position</td><td>2</td></tr>
<tr><td>Newton-Raphson Method</td><td>2</td><td rowspan="2">1</td></tr>
<tr><td>Revision and conclusion</td><td>1</td></tr>
<tr><td colspan="3">UNIT-II: Interpolation</td></tr>
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Relation between operators	1	
Newton's Forward Difference formula for Interpolation	1	1
Newton's Backward Difference formula for Interpolation	1	
Lagrange's Interpolation Formula	2	
Review and conclusion	1	
UNIT–III: Numerical differentiation and integration:		
Introduction	1	1
Derivative using Newton forward Difference formula	2	
Derivative using Newton backward Difference formula	2	
Trapezoidal Rule	1	1
Simpson's $\frac{1}{3}$rd Rule	1	
Simpson's $\frac{3}{8}$th Rule	1	
Review and conclusion	1	
UNIT-IV: First order ordinary Differential Equations		
Exact D.E	2	1
Non-exact D.E	4	
Applications: Newton's law of cooling	2	1
Orthogonal trajectory	2	
UNIT-V: **Higher order linear ordinary differential equations with constant coefficients**		
Solving homogeneous D.E	2	1
Finding Particular integral of Non-Homogenous D.E. when RHS is e^{ax}	2	

Finding Particular integral of Non-Homogenous D.E. when RHS is Sin bx or Cos bx.	2	
Finding Particular integral of Non-Homogenous D.E. when RHS is a polynomial in x.	2	1
Finding Particular integral of Non-Homogenous D.E. when RHS is e^{ax}.(a function of x)	2	
Finding Particular integral of Non-Homogenous D.E. when RHS is x.(a function of x)	2	
UNIT-VI: Partial differentiation		
Total derivative	1	1
Chain rule	1	
Jacobian	1	
Maxima and Minima of functions of 2 or 3 variables with constraints	3	1
Maxima and Minima of functions of 2 or 3 variables without constraints	2	

10. URLs and other e-learning resources

So net CDs &IIT CDs on some of the topics are available in the Digital library.

11. Digital Learning Materials:

- http://nptel.ac.in/courses/106106094
- http://nptel.ac.in/courses/106106094/40
- http://nptel.ac.in/courses/106106094/30
- http://nptel.ac.in/courses/106106094/32
- http://textofvideo. nptl.iitm.ac.in/106106094/lecl.pdf

12. Seminars / group discussions, if any and their schedule: Nil

NUMERICAL METHODS AND DIFFERENTIAL EQUATIONS

UNIT-I
Algebraic and Transcendental Equations
Learning Material

Objectives:

Student should be able to

- Know about the algebraic and Transcendental Equations.
- Understand the Bisection method , method of False Position and Newton Raphson Method.

Syllabus:
Solution of Algebraic and Transcendental Equations- Introduction – Bisection Method – Method of False Position – Newton-Raphson Method.

Learning Outcomes:

Students will be able to

- Solve an Algebraic and Transcendental equation using Numerical Methods

Solutions of Algebraic and Transcendental equations

Introduction : A problem of great importance in science and engineering is that of determining the roots/ zeros of an equation of the form f(x) = 0

- Polynomial function: A function f(x) is said to be a polynomial function if f(x) is a polynomial in x.
i.e. f(x) $= a_0x^n + a_1x^{n-1} + \dots\dots\dots\dots + a^{n-1}x + a_n$ where $a_0 \neq 0$, the coefficients $a_0, a_1 \dots\dots\dots\dots a_n$ are real constants and n is a non-negative integer.
- Algebraic function: A function which is a sum (or) difference (or) product of

two polynomials is called an algebraic function. Otherwise, the function is called a transcendental (or) non-algebraic function.

Eg: $f(x) = c_1 e^x + c_2 e^{-x}$

$$f(x) = e^{5x} - \frac{x^3}{2} + 3$$

- Algebraic Equation: If f(x) is an algebraic function, then the equation f(x) =0 is called an algebraic equation.
- Transcendental Equation: An equation which contains polynomials, exponential functions, logarithmic functions and Trigonometric functions etc. is called a Transcendental equation.

  Ex:- $xe^{2x} - 1 = 0$, $\cos x - x e^x = 0$, $\tan x = x$ are transcendental equations.
- Root of an equation: A number α is called a root of an equation f(x) = 0 if f(α) =0.

  we also say that α is a zero of the function.

  Note: (1) The roots of an equation are the abscissas of the points where the graph y = f(x)

  cuts the x-axis.

  (2) A polynomial equation of degree n will have exactly n roots, real or complex,simple or multiple. A transcendental equation may have one root or infinite number of roots depending on the form of f(x).

Methods for solving the equation

•Direct method:

We know the solution of the polynomial equations such as linear equation $ax + b = 0$ and quadratic equation $ax^2 + bx + c = 0$, will be obtained using direct methods or analytical methods. Analytical methods for the solution of cubic and quadratic equations are also well known to us .

There are no direct methods for solving higher degree algebraic equations or equations involving transcendental functions. such equations are solved by numerical methods.

In these methods we find a interval in which the root lies.

We use the following theorem of calculus to determine an initial approximation. It is also called the Intermediate value theorem.

•**Intermediate value theorem :** If f(x)is continuous on some interval [a, b]and f(a)f(b)< 0, then the equation f(x) = 0 has at least one real root in the interval (a, b).

In this unit we will study some important methods of solving algebraic and transcendental equations.

•**Bisection method**: Bisection method is a simple iteration method to solve an equation. This method is also known as" Bolzano method of successive bisection". Sometimes it is referred to as" Half-interval method". Suppose we know an equation of the form $f(x)=0$ has exactly one real root between two real numbers x_0, x_1 .The number is chosen such that $f(x_0)$ and $f(x_1)$will have opposite sign. Let us bisect the interval $[x_0, x_1]$ into two half intervals and find the midpoint $x_2 = \frac{x_0 + x_1}{2}$. If $f(x_2)=0$ then x_2 is a root.

If $f(x_1)$ and $f(x_2)$ have same sign then the root lies between x_0 and x_2. The interval is taken as (x_0, x_2) Otherwise the root lies in the interval $[x_2, x_1]$.

Repeating the process of bisection , we obtain successive subintervals which are smaller. At each iteration, we get the mid-point as a better approximation of the root. This process is terminated when interval is smaller than the desired accuracy.

Problems:-1) **Find a root of the equation** $x^3 - 5x + 1 = 0$ **using the bisection method in**

5 – stages

Sol: Let $f(x) = x^3 - 5x + 1$

we note that f(0) > 0 and f(1)< 0

$\therefore$ Root lies between 0 and 1

Consider $x_0 = 0 \ \ and \ \ x_1 = 1$

By bisection method the next approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1}{2}(0+1) = 0.5$$

$$\Rightarrow f(x_2) = f(0:5) = -1.375 < 0 \ \ and \ \ f(0) > 0$$

We have the root lies between 0 and 0.5

Now $x_3 = \dfrac{0+0.5}{2} = 0.25$

We find $f(x_3) = -0.234375 < 0 \quad and \quad f(0) > 0$

Since $f(0) > 0$, we conclude that root lies between $x_0 \ \ and \ \ x_3$

The third approximation of the root is

$$x_4 = \frac{x_0 + x_3}{4} = \frac{1}{2}(0+0.25)$$
$$= 0.125$$

We have $f(x_4) = 0.37495 > 0$

Since $f(x_4) > 0 \ and \ f(x_3) < 0$, the root lies between

$x_4 = 0.125 \ \ and \ \ x_3 = 0.25$

Considering the 4th approximation of the roots

$$x_5 = \frac{x_3 + x_4}{2} = \frac{1}{2}(0.125 + 0.25) = 0.1875$$

$f(x_5) = 0.06910 > 0$,

since $f(x_5) > 0 \ \ and \ \ f(x_3) < 0$ the root must lie between $x_5 = 0.18758$ and

$x_3 = 0.25$

Here the fifth approximation of the root is

$$x_6 = \frac{1}{2}(x_5 + x_3)$$
$$= \frac{1}{2}(0.1875 + 0.25)$$
$$= 0.21875$$

We are asked to do up to 5 stages

We stop here 0.21875 is taken as an approximate value of the root and it lies between 0 and 1

False Position Method (Regula – Falsi Method)

In the false position method we will find the root of the equation $f(x)=0$. Consider two initial approximate values x_0 *and* x_1 near the required root so that $f(x_0)$ *and* $f(x_1)$ have different signs. This implies that a root lies between x_0 *and* x_1. The curve $f(x)$ crosses x- axis only once at the point x_2 lying between the points x_0 *and* x_1, Consider the point $A=(x_0, f(x_0))$ and $B=(x_1, f(x_1))$ on the graph and suppose they are connected by a straight line, Suppose this line cuts x-axis at x_2, We calculate the values of $f(x_2)$ at the point. If $f(x_0)$ *and* $f(x_2)$ are of opposite sign, then the root lies between x_0 *and* x_2 and value x_1 is replaced by x_2

Otherwise the root lies between x_2 and x_1 and the value of x_0 is replaced by x_2

Another line is drawn by connecting the newly obtained pair of values. Again the point here the line cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points x_2, x_3, x_4 .....obtained converge to the expected root of the equation $y=f(x)$

To obtain the equation to find the next approximation to the root

Let $A=(x_0, f(x_0))$ *and* $B=(x_1, f(x_1))$ be the points on the curve $y=f(x)$ Then the equation to the chord AB is $\frac{y-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x-x_0} \rightarrow (1)$

At the point C where the line AB crosses the x – axis, we have f(x) =0 i.e. y =0

From (1), we get $x = x_0 - \frac{x_1 - x_0}{f(x_1)-f(x_0)} f(x_0) \rightarrow (2)$

x is given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new values of x is taken as x_2 then (2) becomes

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{---------------3}$$

Now we decide whether the root lies between

$x_0 \ and \ x_2 \ (or) x_2 \ and \ x_1$

We name that interval as (x_1, x_2) The line joining $(x_1, y_1)(x_2, y_2)$ meets x – axis at x_3 is given by $x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

This will in general, be nearest to the exact root we continue this procedure till the root is found to the desired accuracy

The iteration process based on (3) is known as the method of false position

The successive intervals where the root lies, in the above procedure are named as

$(x_0, x_1), (x_1, x_2), (x_2, x_3)$ etc

Where $x_i < x_i + 1$ and $f(x_0), f(x_i + 1)$ are of opposite signs

Also $x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

Problems:-

1.Find out the roots of the equation $x^3 - x - 4 = 0$ using false position method

sol: Let $f(x) = x^3 - x - 4 = 0$

$f(0) = -4, f(1) = -4, f(2) = 2$

Since $f(1) \ and \ f(2)$ have opposite signs the root lies between 1 and 2

By false position method $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$x_2 = \frac{(1\times 2) - 2(-4)}{2-(-4)}$$

$$= \frac{2+8}{6} = \frac{10}{6} = 1.666$$

$$f(1.666) = (1.666)^3 - 1.666 - 4$$
$$= -1.042$$

Now, the root lies between 1.666 and 2

$$x_3 = \frac{1.666\times 2 - 2\times(-1.042)}{2-(-1.042)} = 1.780$$

$$f(1.780) = (1.780)^3 - 1.780 - 4$$
$$= -0.1402$$

Now, the root lies between 1.780 and 2

$$x_4 = \frac{1.780\times 2 - 2\times(-0.1402)}{2-(-0.1402)} = 1.794$$

$$f(1.794) = (1.794)^3 - 1.794 - 4$$
$$= -0.0201$$

Now, the root lies between 1.794 and 2

$$x_5 = \frac{1.794\times 2 - 2\times(-0.0201)}{2-(-0.0201)} = 1.796$$

$$f(1.796) = (1.796)^3 - 1.796 - 4 = -0.0027$$

Now, the root lies between 1.796 and 2

$$x_6 = \frac{1.796\times 2 - 2\times(-0.0027)}{2-(-0.0027)} = 1.796$$

The root is 1.796

Newton- Raphson Method:-

The Newton- Raphson method is a powerful and elegant method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let x_0 be an approximate root of $f(x)=0$ and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1)=0$.

By Taylor's theorem neglecting second and higher order terms

$$f(x_1)=f(x_0+h)=0$$

$$\Rightarrow f(x_0)+hf^1(x_0)=0$$

$$\Rightarrow h=-\frac{f(x_0)}{f^1(x_0)}$$

Substituting this in x_1 we get

$$x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$\therefore x_1$ is a better approximation than x_0

Successive approximations are given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

Problem:- **1**.Find by Newton's method, the real root of the equation $xe^x - 2 = 0$ Correct to three decimal places.

Sol. Let $f(x)=xe^x-2 \rightarrow (1)$

Then $f(0)=-2$ and $f(1)=e-2=0.7183$

So root of $f(x)$ lies between 0 and 1

It is near to 1. so we take $x_0=1$ and $f^1(x)=xe^x+e^x$ *and* $f^1(1)=e+e=5.4366$

$\therefore$ By Newton's Rule

First approximation $x_1 = x_0 - \dfrac{f(x_0)}{f^1(x_0)}$

$$= 1 - \frac{0.7183}{5.4366} = 0.8679$$

$\therefore f(x_1) = 0.0672 \qquad f^1(x_1) = 4.4491$

The second approximation $x_2 = x_1 - \dfrac{f(x_1)}{f^1(x_1)}$

$$= 0.8679 - \frac{0.0672}{4.4491}$$
$$= 0.8528$$

$\therefore$ Required root is 0.853 correct to 3 decimal places.

Convergence of the Iteration Methods

We now study the rate at which the iteration methods converge to the exact root, if the initial approximation is sufficiently close to the desired root.
Define the error of approximation at the k th iterate as $\epsilon_k = x_k - \alpha$, k = 0, 1, 2,...

Definition: An iterative method is said to be of order p or has the rate of convergence p, if p is the largest positive real number for which there exists a finite constant C≠0 , such that

$$|\epsilon_{k+1}| \square | \in_k^p |$$

The constant C, which is independent of k, is called the asymptotic error constant and it depends on the derivatives of f(x) at x = α.

Assignment-cum-Tutorial Questions

UNIT-I

SECTION-A

Objective Questions

1. Every algebraic equation of nth degree has exactly --------- roots.
2. In bisection method if root lies between a and b then f(a).f(b) --------------
3. A root of $x^3 - x + 1 = 0$ lies between …………………
4. Newton -Raphson method fails when ----------------------------
5. If first approximation of roots $x^2 - x - 4 = 0$ is $x_0 = 2$ then x_1 by Newton Raphson method is……………..
6. Newton's iterative formula to find the value of $3\sqrt{N}$ is……………
7. If first two approximations of root of $xe^x - 3 = 0$ are 1 and 1.5 then x_2 by regula flasi method is [ ]

   a) 1.21 b) 1.425 c) 1.035 d) 1.312

8. If first two approximations $x_0\ and\ x_1$ of root of $x^3 - x^2 - 2 = 0$ are 1.5 and 2 then x_2 by regula falsi method is [ ]

   a) 1.652 b) 1.724 c) 1.892 d) 1.928

9. If $x_0\ and\ x_1$ are 1.4 and 1.5 by false position method find x_2 for $x^2 - 1 - \sin x = 0$ [ ]

   a) 1.0009 b) 1.2097 c) 1.1940 d) 1.4091

10. If first two approximations $x_0\ and\ x_1$ for the root of $x^3 - 3x - 4 = 0$ are 2.125 and -3 then x_2 by regula- falsi method is [ ]

    a) -2.521 b) -2.34 c) -2.171 d) -2.79

11. The formula to find $(n+1)^{th}$ approximation of root of f(x) = 0 by Newton Raphson method is [ ]

    a) $x_{n+1} = x_n - \dfrac{f(x)}{f(x_{n+1})}$ b) $x_{n+1} = x_n - \dfrac{f^1(x_n)}{f(x_n)}$

c) $x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$ d) $x_{n+1} = x_n \frac{f(x_n)}{f^1(x_n)}$

SECTION-B

Subjective Questions

1. Find out the roots of the equation $x^3 - x - 4 = 0$ by False position method
2. Derive the formula for Newton-Raphson Method .
3. Write a short notes on Bisection method.
4. Explain the procedure involved in finding the solution by Regula-Falsi method.
5. Find a positive real root of f(x) = cosx + 1 - 3x = 0 correct to two decimal places by bisection method
6. Find the positive root of the following equation by the method of interval halving for $x^3+x-1 = 0$
7. Using Newton – Raphson method
8. Find square root and cube root of a number N
9. Find reciprocal of a number
10. If [a, b] is the initial guess interval and if f(a) and f(b) are the function values at x=a & x=b, then derive that the approximated root is given by $x = \frac{af(b) - bf(a)}{f(b) - f(a)}$.
11. Find an approximate root of $x \log_{10}^{x} - 1.2 = 0$ by Regula- falsi method
12. Find a positive root of the equation 3x=cosx+1 by Newton-Raphson Method
13. Find a real root of $xe^x - \cos x = 0$ by Newton-- Raphson method
14. Find a root of the following equation using the Bisection method correct to three decimal places: $x^2 - 4x - 9 = 0$.

SECTION-C

GATE Questions:

1. The Newton-Raphson method is used to solve the equation $f(x) = x^3 - 5x^2 + 6x - 8 = 0$. Taking the initial guess as $x = 5$, the solution obtained at the end of the first iteration is _____.2015

2. A numerical solution of the equation $f(x)= x + \sqrt{x} - 3 = 0$ can be obtained using Newton- Raphson method. If the starting value is x = 2 for the iteration, the value of x that is to be used in the next step is---2011

a) 0.306 b) 0.739 c)1.694 d)2.306 [ ]

3. The recursion relation to solve $x=e^{-x}$ using Newton Raphson method is----2008

4. The equation $x^3 - x^2 + 4x - 4 = 0$ is to be solved using the Newton-Raphson method. If $x = 2$ is taken as the initial approximation of the solution, then the next approximation using this method will be-----

a) $\frac{2}{3}$ b) $\frac{3}{2}$ c) $\frac{4}{3}$ d) 1 [ ]

5. Newton-Raphson method is used to compute a root of the equation $x^2-13=0$ with 3.5 as the initial value. The approximation after one iteration is----------------

a) 3.676 b) 3.667 c) 3.607 d) 3.575 [ ]

6. The Newton-Raphson iteration $x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$ can be used to compute the-------------2008 [ ]

a) Square of R b) Reciprocal of R

c) Square root of R d) Logarithm of R

7. The Newton-Raphson method is used to solve the equation $f(x) = x^3 - 5x^2 + 6x - 8 = 0$. Taking the initial guess as $x = 5$, the solution obtained at the end of the first iteration is _____.2015

8. The real root of the equation $5x - 2\cos x - 1 = 0$ (up to two decimal accuracy) is _______2014

NUMERICAL METHODS AND DIFFERENTIAL EQUATIONS

UNIT-II

INTERPOLATION

Objectives:

- Develop an understanding of the use of numerical methods in modern scientific computing.
- To gain the knowledge of Interpolation

Syllabus:

Interpolation- Introduction – Finite differences- Forward Differences – Backward differences –Central differences – Symbolic relations – Newton formulae for interpolation – Lagrange's interpolation..

Learning Outcomes:

Student should be able to

- Know about the Interpolation, and Finite Differences.
- Utilize the Newton's formula for interpolation
- Operate Lagrange's Interpolation formula..

Learning Material

UNIT-II

Interpolation

Introduction:-

consider the statement y =f(x) , $x_0 \le x \le x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0 \le x \le x_n$. If the function f(x) is single valued and continuous and is known explicitly then the values of f(x) for certain values of x like $x_0, x_1, \ldots\ldots x_n$ can be calculated. The problem now is if we are given the set of tabular values

$$x : x_0 \quad x_1 \quad x_2 \ldots\ldots x_n$$
$$y : y_0 \quad y_1 \quad y_2 \ldots\ldots y_n$$

Satisfying the relation y = f(x) and the explicit definition of f(x) is not known, is it possible to find a simple function say f(x) such that f(x) and $\phi(x)$ agree at the set of tabulated points. This process of finding $\phi(x)$ is called interpolation. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation

Finite Differences:-

1. **Introduction**:- Here we introduce forward, backward and central differences of a function y = f(x). These differences play a fundamental role in the study of differential calculus, which is an essential part of numerical applied mathematics

2. Forward Differences:-

Consider a function y = f(x) of an independent variable x. let $y_0, y_1, y_2 \ldots\ldots y_r$ be the values of y corresponding to the values $x_0, x_1, x_2 \ldots\ldots x_r$ of x respectively. Then the differences $y_1 - y_0$, $y_2 - y_1 \ldots\ldots\ldots$ are called the first forward differences of y, and we denote them by $\Delta y_0, \Delta y_1, \ldots\ldots$ that is

$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2 \ldots\ldots$$

In general $\Delta y_r = y_{r+1} - y_r \therefore r = 0,1,2 - - - - -$

Here the symbol Δ is called the forward difference operator

The second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1 \ldots\ldots$ that is

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

In general $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad r = 0,1,2 \ldots\ldots$ similarly, the n^{th} forward differences are defined by the formula.

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \quad r = 0,1,2 \ldots\ldots$$

The symbol Δ^n is referred as the n^{th} forward difference operator.

3. Forward Difference Table:-

The forward differences are usually arranged in tabular columns as shown in the following table called a forward difference table

Values of x	Values of y	First order differences	Second order differences	Third order differences	Fourth order differences
x_o	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
x_4	y_4	$\Delta y_3 = y_4 - y_3$			

4. Backward Differences:-

Let $y_0, y_1 ......y_r......$ be the values of a function $y = f(x)$ corresponding to the values $x_0, x_1, x_2.............x_r......$ of x respectively. Then, $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2,....$ are called the first backward differences

In general $\nabla y_r = y_r - y_{r-1},\ r = 1,2,3......... \rightarrow (1)$

The symbol ∇ is called the backward difference operator, like the operator Δ, this operator is also a linear operator

Comparing expression (1) above with the expression (1) of section we immediately note that $\nabla y_r = \nabla y_{r-1}, r = 0,1,2....... \rightarrow (2)$

The first backward differences of the first background differences are called second differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3 ---\nabla^2{}_r ----$ i.e.,..

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2 ....$$

In general $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, r = 2,3..... \rightarrow (3)$ similarly, the n^th backward differences are defined by the formula $\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, r = n, n+1..... \rightarrow (4)$

If $y = f(x)$ is a constant function , then y = c is a constant, for all x, and we get $\nabla^n y_r = 0 \forall n$ the symbol ∇^n is referred to as the n^th backward difference operator

5. Backward Difference Table:-

X	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		∇y_1		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\nabla^3 y_3$
x_2	y_2		$\nabla^2 y_3$	
		∇y_3		
x_3	y_3			

6. Central Differences:-

With $y_0, y_1, y_2 .... y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2 ..... x_r .... \; of \; x$, we define the first central differences

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2} ----$ as follows

$$\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \delta y_{5/2} = y_3 - y_2 ----$$

$$\delta y_{r-1/2} = y_r - y_{r-1} \rightarrow (1)$$

The symbol δ is called the central differences operator. This operator is a linear operator

Comparing expressions (1) above with expressions earlier used on forward and backward differences we get

$\delta y_{1/2} = \Delta y_0 = \nabla y_1, \delta y_{3/2} = \Delta y_1 = \nabla y_2 .....$

In general $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}, n = 0,1,2...... \rightarrow (2)$

The first central differences of the first central differences are called the second central differences and are denoted by $\delta^2 y_1, \delta^2 y_2 ...$

Thus $\delta^2 y_1 = \delta_{3/2} - \delta y_{1/2}, \delta^2 y_2 = \delta_{5/2} - \delta_{3/2} ....$

$\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2} \rightarrow (3)$

Higher order central differences are similarly defined. In general the n^th^ central differences are given by

for odd $n : \delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}, r = 1,2.... \rightarrow (4)$

for even $n : \delta^n y_r = \delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2}, r = 1,2.... \rightarrow (5)$

while employing for formula (4) for $n = 1$, we use the notation $\delta^0 y_r = y_r$

If y is a constant function, that is if $y = c$ a constant, then $\delta^n y_r = 0 \;\; for \;\; all \;\; n \geq 1$

7. Central Difference Table

x_0	y_0	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{2/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			

x_4	y_4				

Symbolic Relations :

E-operator:- The shift operator E is defined by the equation $Ey_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives $E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$

Generalizing $E^n y^r = y_{r+n}$

Averaging operator:- The averaging operator μ is defined by the equation $\mu y_r = \frac{1}{2}\left[y_{r+1/2} + y_{r-1/2}\right]$

Relationship Between $\Delta \; and \; E$

We have

$$\begin{aligned}\Delta y_0 &= y_1 - y_0 \\ &= Ey_0 - y_0 = (E-1)\, y_0\end{aligned}$$

$$\Rightarrow \Delta = E - y(or)\, E = 1 + \Delta$$

Some more relations

$$\begin{aligned}\Delta^3 y_0 = (E-1)^3\, y_0 &= \left(E^3 - 3E^2 + 3E - 1\right) y_0 \\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

•**Inverse operator**: Inverse operator E^{-1} is defined as $E^{-1} y_r = y_{r-1}$

In general $E^{-n} y_n = y_{r-n}$

We can easily establish the following relations

i) $\nabla \equiv 1 - E^{-1}$ ii) $\delta \equiv E^{1/2} - E^{-1/2}$ iii) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

iv) $\Delta = \nabla E = E^{1/2}$ v) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

•**Differential operator:**

The operator D is defined as $Dy(x) = \frac{\partial}{\partial x}\left[y(x)\right]$

Relation between the Operators D and E

Using Taylor's series we have, $y(x+h) = y(x) + hy^1(x) + \frac{h^2}{2!}y^{11}(x) + \frac{h^3}{3!}y^{111}(x) + ----$

This can be written in symbolic form

$$Ey_x = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ----\right]y_x = e^{hD}.y_x$$

We obtain in the relation $E = e^{hD} \rightarrow (3)$

•**Theorem:** If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is constant

Note:-

As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1} f(x) = 0, \Delta^{n+2} f(x) = 0, ...$

The converse of above result is also true that is, if $\Delta^n f(x)$ is tabulated at equal spaced intervals and is a constant, then the function $f(x)$ is a polynomial of degree n

1. Find the missing term in the following data

X	0	1	2	3	4
Y	1	3	9	-	81

Why this value is not equal to 3^3. Explain

Sol. Consider $\Delta^4 y_0 = 0$

$\Rightarrow 4y_0 - 4y_3 + 5y_2 - 4y_1 + y_0 = 0$

Substitute given values we get

$81 - 4y_3 + 54 - 12 + 1 = 0 \Rightarrow y_3 = 31$

From the given data we can conclude that the given function is $y=3^x$. To find y_3, we have to assume that y is a polynomial function, which is not so. Thus we are not getting $y=3^3=27$

2. Evaluate

$(i)\Delta\cos x$

$(ii)\Delta^2\sin(px+q)$

$(iii)\Delta^n e^{ax+b}$

Sol. Let h be the interval of differencing

$$(i)\Delta\cos x=\cos(x+h)-\cos x$$

$$=-2\sin\left(x+\frac{h}{2}\right)\sin\frac{\text{h}}{2}$$

$$(ii)\Delta\sin(px+q)=\sin\left[p(x+h)+q\right]-\sin(px+q)$$

$$=2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}$$

$$=2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\Delta^2\sin(px+q)=2\sin\frac{ph}{2}\Delta\left[\sin(px+q)+\frac{1}{2}(\pi+ph)\right]$$

$$=\left[2\sin\frac{ph}{2}\right]^2\sin\left[px+q+\frac{1}{2}(\pi+ph)\right]$$

$$(iii)\Delta e^{ax+b}=e^{a(x+h)+b}-e^{ax+b}$$

$$=e^{(ax+b)}\left(e^{ah-1}\right)$$

$$\Delta^2e^{ax+b}=\Delta\left[\Delta\left(e^{ax+b}\right)\right]-\Delta\left[\left(e^{ah}-1\right)\left(e^{ax+b}\right)\right]$$

$$=\left(e^{ah}-1\right)^2\Delta\left(e^{ax+h}\right)$$

$$=\left(e^{ah}-1\right)^2e^{ax+b}$$

Proceeding on, we get $\Delta^n\left(e^{ax+b}\right)=\left(e^{ah}-1\right)^n e^{ax+b}$

Newton's Forward Interpolation Formula:-

Let $y=f(x)$ be a polynomial of degree n and taken in the following form

$$y=f(x)=b_0+b_1(x-x_0)+b_2(x-x_0)(x-x_1)+b_3(x-x_0)(x-x_1)(x-x_2)+---$$
$$+b_n(x-x_0)(x-x_1)----(x-x_{n-1})\rightarrow(1)$$

This polynomial passes through all the points (for i = 0 to n. Therefore, we can obtain the $y_i's$ by substituting the corresponding $x_i's$ as

$$at\ \ x=x_0, y_0=b_0$$
$$at\ \ x=x_1, y_1=b_0+b_1(x_1-x_0)$$
$$at\ \ x=x_2, y_2=b_0+b_1(x_2-x_0)+b_2(x_2-x_0)(x_2-x_1)\rightarrow(1)$$

Let 'h' be the length of interval such that $x_i's$ represent

$$x_0, x_0+h, x_0+2h, x_0+3h----x_0+xh$$

This implies $x_1-x_0=h, x_2-x_0-2h, x_3-x_0=3h----x_n-x_0=nh\rightarrow(2)$

From (1) and (2), we get

$$y_0=b_0$$
$$y_1=b_0+b_1h$$
$$y_2=b_0+b_1 2h+b_2(2h)h$$
$$y_3=b_0+b_1 3h+b_2(3h)(2h)+b_3(3h)(2h)h$$
....

....
$$y_n=b_0+b_1(nh)+b_2(nh)(n-1)h+----+b_n(nh)\left[(n-1)h\right]\left[(n-2)h\right]\rightarrow(3)$$

Solving the above equations for $b_0, b_{11}, b_2.....b_n$, we get $b_0=y_0$

$$b_1=\frac{y_1-b_0}{h}=\frac{y_1-y_0}{h}=\frac{\Delta y_0}{h}$$
$$b_2=\frac{y_2-b_0-b_1 2h}{2h^2}=y_2-y_0-\frac{(y_1-y_0)}{h}2h$$

$$= \frac{y_2 - y_0 - 2y_1 - 2y_0}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

$$\therefore b_2 = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly, we can see that

$$b_3 = \frac{\Delta^3 y_0}{3!h^3}, b_4 = \frac{\Delta^4 y_0}{4!h^4} ------ b_n = \frac{\Delta^n y_0}{n!h^n}$$

$$\therefore y = f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1)$$

$$+ \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + ---- +$$

$$+ \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1) --- (x - x_{n-1}) \rightarrow (3)$$

If we use the relationship $x = x_0 + ph \Rightarrow x - x_0 = ph,\ where\ p = 0,1,2,.....n$

Then

$$x - x_1 = x - (x_0 + h) = (x - x_0) - h$$

$$= ph - h = (p-1)h$$

$$x - x_2 = x - (x_1 + h) = (x - x_1) - h$$

$$= (p-1)h - h = (p-2)h$$

....

$$x - x_i = (p - i)h$$

....

$$x - x_{n-1} = [p - (n-1)]h$$

Equation (3) becomes

$$y = f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ----- +$$

$$\frac{p(p-1)(p-2)----(p-(n-1))}{n!}\Delta^n y_0 \rightarrow (4)$$

Newton's Backward Interpolation Formula:-

If we consider

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + -----(x - x_i)$$

and impose the condition that y and $y_n(x)$ should agree at the tabulated points $x_n, x_n - 1, \ldots\ldots x_2, x_1, x_0$

We obtain

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2i}\nabla^2 y_n + ---$$

$$\frac{p(p+1)----[p+(n-1)]}{n!}\nabla^n y_n + ----\rightarrow(6)$$

Where $p = \frac{x - x_n}{h}$

This uses tabular values of the left of y_n. Thus this formula is useful formula is useful for interpolation near the end of the tabular values

Q:-1.Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula

Percentage of lead(p)	50	60	70	80
Temperature $(Q°c)$	205	225	248	274

Sol. The difference table is

x	Y	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

Let temperature = $f(x)$

$$x_0 + ph = 24, x_0 = 50, h = 10$$
$$50 + p(10) = 54 \ (or)\ p = 0.4$$

By Newton's forward interpolation formula

$$f(x_0+ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{n!}\Delta^3 y_0 + ---$$
$$f(54) = 205 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(3) + \frac{(0.4)(0.4-1)(0.4-2)}{3!}(0)$$
$$= 205 + 8 - 0.36$$
$$= 212.64$$

Melting point = 212.6

2.Using Newton's Gregory backward formula, find *e* $^{1.9}$ from the following data

x	**1.00**	**1.25**	**1.50**	**1.75**	**2.00**
e^{x}	**0.3679**	**0.2865**	**0.2231**	**0.1738**	**0.1353**

Lagrange's Interpolation Formula:-

Let $x_0, x_1, x_2, \ldots.\ x_n$ be the $(n+1)$ values of x which are not necessarily equally spaced. Let $y_0, y_1, y_2 \ldots\ldots.. y_n$ be the corresponding values of $y = f(x)$ let the polynomial of degree n for the function $y = f(x)$ passing through the $(n+1)$ points $(x_0, f(x_0)), (x_1, f(x_1)) ---- (x_n, f(x_n))$ be in the following form

$$y = f(x) = a_0(x-x_1)(x-x_2)\ldots\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\ldots\ldots\ldots(x-x_n) +$$
$$a_2(x-x_0)(x-x_1)\ldots\ldots.(x-x_n) + \ldots\ldots.. + a_n(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1}) \rightarrow (1)$$

Where $a_0, a_1, a_2 \ldots.$ a^n are constants

Since the polynomial passes through $(x_0, f(x_0))$, $(x_1, f(x_1))......(x_n, f(x_n))$. The constants can be determined by substituting one of the values of $x_0, x_1, ...x_n \ for \ x$ in the above equation

Putting $x = x_0$ in (1) we get, $f(x_0) = a_0(x - x_1)(x_0 - x_2)(x_0 - x_n)$

$$\Rightarrow a_0 = \frac{f(x_0)}{(x - x_1)(x_0 - x_2)....(x_0 - x_n)}$$

Putting $x = x_1$ in (1) we get, $f(x_1) = a_1(x - x_0)(x_1 - x_2) ---- (x_1 - x_n)$

$$\Rightarrow a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)....(x_1 - x_n)}$$

Similarly substituting $x = x_2$ in (1), we get

$$\Rightarrow a_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)......(x_2 - x_n)}$$

Continuing in this manner and putting $x = x_n$ in (1) we get

$$a_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) ---- (x_n - x_{n-1})}$$

Substituting the values of $a_0, a_1, a_2....a_n$, we get

$$f(x) = \frac{(x - x_1)(x - x_2).......(x - x_n)}{(x_0 - x_1)(x_0 - x_2).........(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2).....(x - x_n)}{(x_1 - x_0)(x_1 - x_2)....(x_1 - x_n)}$$

$$f(x_1) + \frac{(x - x_0)(x - x_1)(x - x_2).....(x - x_n)}{(x_2 - x_0)(x_2 - x_1)......(x_2 - x_n)} + ...f(x_2) + \frac{(x - x_0)(x - x_1).....(x - x_{n-1})}{(x_n - x_1)(x_n - x_2).....(x_n - x_{n-1})} f(x_n)$$

Q 1. Using Lagrange's formula calculate $f(3)$ from the following table

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Sol. Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_5 = 6, x_4 = 5$

$$f(x_0) = 1, f(x_1) = 14, f(x_2) = 15, f(x_3) = 5, f(x_4) = 6, f(x_5) = 19$$

From lagrange's interpolation formula

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}f(x_0)$$
$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}f(x_2)$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}f(x_5)$$

Here $x=3$ then

$$f(3)=\frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}\times 1+$$
$$\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}\times 14+$$
$$\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}\times 15+$$
$$\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}\times 5+$$
$$\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)}\times 6+$$
$$\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}\times 19$$

$$=\frac{12}{240}-\frac{18}{60}\times 14+\frac{36}{48}\times 15+\frac{36}{48}\times 5-\frac{18}{60}\times 6+\frac{12}{40}\times 19$$

$$=0.05-4.2+11.25+3.75-1.8+0.95$$

$$=10$$

$f(x_3) = 10$

Assignment-Cum-Tutorial Questions
UNIT-II
SECTION-A

Objective Questions

1. The following is used for unequal interval of x values [ ]
   a) Lagrange's formula
   b) Newton's forward
   c) Newton's backward interpolation formula
   d) Gauss forward interpolation formula

2. The (n+1)th order difference a polynomial of n^{th} degree is [ ]
   a) polynomial of n^{th} degree b) zero
   c) polynomial on first degree d) constant

3.

X	1	2	3	4
f(x)	1	4	27	64

If x=2.5 then p=
a) 1.5 b) 1 c) 2.5 d) 2 [ ]

4.

X	0.1	0.2	0.3	0.4
f(x)	1.005	1.02	1.045	1.081

When p=0.6, x=
a) 0.16 b) 0.26 c) 0.1 d) 3.0 2. [ ]

5. Relation between Backward and Shifting operator is ________________.
6. When do we apply Lagrange's interpolation?
7. Say True or False:

Newton's Interpolation formulae are not suited to estimate the value of a function near the middle of the table.

8. If $y = x^2 + 2x$ then $\Delta^3 y =$ [ ]

a) 1 b) 2 c) 0 d) 3

9. $\frac{\Delta^2}{E}\left(e^x\right) = \_\_\_\_\_\_$ [ ]

a) $e^x\left(e^h - 1\right)^2$ b) $e^x\left(e^h - 1\right)$ c) $e^{x-h}\left(e^h - 1\right)^2$ d) $e^x\left(e^{h-1}\right)$

10. $\left(E^{1/2} + E^{-1/2}\right)\left(1+\Delta\right)^{1/2}$

a) $1+\Delta$ b) $2+\Delta$ c) $1-\Delta$ d) Δ [ ]

11. $\frac{\delta^2}{4} + 1 =$ [ ]

a) μ b) μ^2 c) $\mu + \Delta$ d) $\Delta - 1$

12.

X	0	1	2
F(x)	7	10	13

By Newton's forward formula f(2.5)=

a) 15.25 b) 16.75 c) 16.25 d)16.1 [ ]

SECTION – B

Subjective Questions

1. Certain corresponding values of x and log x are (300, 2.4771), (304, 2.4829), (305, 2.4843) and (307, 2.4871). Find log 301.

2. Find a cubic polynomial in x which takes on the values -3, 3, 11, 27, 57 and 107, when x=0, 1, 2, 3, 4 and 5 respectively.

3. Using Newton's forward interpolation formula, for the given table of values

X	1.1	1.3	1.5	1.7	1.9
$f(x)$	0.21	0.69	1.25	1.89	2.61

Obtain the value of f(x) when x= 1.4.

4. The population of a town in the decimal census was given below. Estimate the population for the 1895

Year x	1891	1901	1911	1921	1931
Population of y	46	66	81	93	101

5. Find the cubic polynomial which takes the values y(0)=1,y(1)=0,y(2)=1,y(3)=10

6. Using Newton's backward formula find the value of sin38 ?

x:	0	10	20	30	40
sin x:	0	.17365	.34202	.50000	.64279

7. Fit a polynomial of degree three which takes the following values

x:	3	4	5	6
y:	6	24	60	120

8. Using Newton's forward formula, find the value of f(1.6) if

X	1	1.4	1.8	2.2	2.6
y	3.49	4.82	5.96	6.5	8.4

9. Find log 58.75 from the following data:

X	40	45	50	55	60	65
log x	1.60206	1.65321	1.69897	1.74036	1.77815	1.81291

Using Newton's Backward Interpolation formula.

10. Find the Lagrange's interpolating polynomial and using it find y when x = 10, if the values of x and y are given as follows:

x	5	6	9	11
Y	12	13	14	16

11. Prove that $\Delta^{10}\left[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)\right] = 24X2^{10}X10!$ if h=2.

12. Find y(42) from the following data using Newton's interpolation formula

X	20	25	30	35	40	45
Y	354	332	291	260	231	204

13. Using Lagrange's formula to fit a polynomial to the data and hence find y(1).

X	-1	0	2	3
Y	-8	3	1	12

14. Find the number of students who got marks between 40 and 45

Marks	:	30-40	40-50	50-60	60-70	70-80
No. of students	:	31	42	51	35	31

15. The area A of a circle of diameter d is given below:

d:	80	85	90	95	100
A:	5026	5674	6362	7088	7854

Find approximately the areas of the circles of diameters 82 and 91.

SECTION-C

1. Evaluate $\Delta^{10}(1-x)(1-2x)(1-3x)..........(1-10x)$ taking h=1

NUMERICAL METHODS AND DIFFERENTIAL EQUATIONS

UNIT-III

Numerical differentiation and integration

Learning Material

Objectives:

- To understand the concepts of numerical differentiation and integration.

Syllabus:

Approximation of derivative using Newton's forward and backward formulas. Integration using Trapezoidal and Simpson's rules.

Learning Outcomes:

At the end of the unit, Students will be able to

- Calculate the area and slope of a given curve.

Learning Material

Numerical solutions of ordinary Differential Equations

Introduction:-

Suppose a function y = f(x) is given by a table of values (x_i, y_i). The process of computing the derivative $\frac{dy}{dx}$ for some particular value of x is called Numerical differentiation.

Derivatives using Newton's forward difference formula

Newton's interpolation formula for equal intervals is

Suppose that we are given a set of values (x_i, y_i), i = 0,1,2,...., n.

We want to find the derivative of y = f (x) passing through the (n + 1)points, at a point nearer to the starting value at x = x_0.

Newton's Forward Difference Interpolation Formula is

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots \quad \text{-------------------} \ (1)$$

$$\text{Where } p = \frac{x - x_0}{h} \quad \text{------------------} \ (2)$$

On differentiation (1) w.r.t., p we have

On differentiation (2) w.r.t. x we have, $\frac{dp}{dx} \approx \frac{1}{h}$

$$\frac{dy}{dx} = \frac{dy}{dp}.\frac{dp}{dx} = \frac{1}{\mathrm{h}}\left[\begin{array}{l}\Delta \mathrm{y}0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 \\ + \frac{4p^3-18p^2+22p-6}{24}\Delta^4 y_0 + \ldots\end{array}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Equation (3) gives the value of $\frac{dy}{dx}$ at any point x which may be anywhere in the interval.

At x = x_0 and p = 0, hence putting p = 0, equation (3) gives

$$\left(\frac{dy}{dx}\right)_{x \approx x_1} = \left(\frac{dy}{dp}\right)_{p \approx 1} = \frac{1}{\mathrm{h}}\left|\begin{array}{l}\Delta \mathrm{y}0 + \frac{1}{2}\Delta^2 y_0 + \frac{1}{6}\Delta^3 y_0 \\ + \frac{4p^3-18p^2+22p-6}{24}\Delta^4 y_0 + \ldots\end{array}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Again on differentiation (3) we get

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\mathrm{d}}{\mathrm{dp}}\left(\frac{dy}{dx}\right).\frac{dp}{dx} = \frac{\mathrm{d}}{\mathrm{dp}}\left(\frac{dy}{dx}\right).\frac{dp}{dx} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(p-1)}{}\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \ldots\right]$$

From which we obtain

$$\left(\frac{d^2y}{dx^2}\right) x \approx x_0 = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + ..\right] \text{ at x = } x_0 \text{ and p = 0 } \ldots\ldots\ (5)$$

Similarly, $$\left(\frac{d^3y}{dx^3}\right) x \approx x_0 = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\ldots\right] \qquad \ldots\ldots\ (6)$$

Derivatives using Newton's Backward Difference Formula:

Newton's Backward Difference Interpolation Formula is

$$y(x) = y_n + p\,\Delta y_n + \frac{p(p+1)}{2!}\Delta^2 y_n + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_n + \ldots\ldots\ldots\ldots \quad \text{------------------ (7)}$$

$$\text{Where } p = \frac{x - x_n}{h} \quad \text{------------------ (8)}$$

On differentiation (7) w.r.t., p we have

$$\frac{dy}{dp} = \left[\Delta y_n + \frac{2p+1}{2}\Delta^2 y_n + \frac{3p^2+6p+2}{6}\Delta^3 y_n + \frac{4p^3+18p^2+22p+6}{24}\Delta^4 y_n + \ldots\right].$$

On differentiation (8) w.r.t. x we have, $\frac{dp}{dx} \approx \frac{1}{h}$ Now

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2}\Lambda^2 y_n + \frac{3p^2+6p+2}{6}\Lambda^3 y_n + \frac{4p^3+18p^2+22p+6}{24}\Lambda^4 y_n + ..\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Equation (9) gives the value of $\frac{dy}{dx}$ at any point x which may be anywhere in the interval.

At $x = x_n$ and p = 0, hence putting p = 0, equation (9) gives

$$\left(\frac{dy}{dx}\right)x \approx x_{n_1} = \left(\frac{dy}{dx}\right)x_n = \frac{1}{h}\left[\Delta y n + \frac{1}{2}\Delta^2 y_n + \frac{1}{3}\Delta^3 y_n + \frac{1}{4}\Delta^4 y_n + \ldots\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Again on differentiation (09) we obtain

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx}\cdot\frac{dp}{dx} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\cdot\frac{dp}{dx} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\cdot\frac{dp}{dx}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_n + \frac{(p+1)}{}\Delta^3 y_n + \frac{6p^2+18p+11}{12}\Delta^4 y_n + .\right]$$

From which we obtain

$$\left(\frac{d^2y}{dx^2}\right) x \approx x_n = \frac{1}{h^2}\left[\Delta^2 y_n + \Delta^3 y_n + \frac{11}{12}\Delta^4 y_n + \frac{5}{6}\Delta^5 y_n + ..\right] \text{ at } x = x_n \text{ and } p = 0$$

Similarly, $$\left(\frac{d^3y}{dx^3}\right) x \approx x_n = \frac{1}{h^3}\left[\Delta^3 y_n - \frac{3}{2}\Delta^4 y_0 + ...\right] \qquad .... (12)$$

Problem 1. Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at* $x = 51$ from the following data.

x	50	60	70	80	90
y	19.96	36.65	58.81	77.21	94.61

Solution: Here h = 10. To find the derivatives of y at x = 51 we use Newton's Forward difference formula taking the origin at a = 50.

We have $p = \frac{x - x_0}{h} = \frac{51-50}{10} 0.1$

$$\left(\frac{dy}{dx}\right)_{x=51} = \left(\frac{dy}{dx}\right)_{p=0.1} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{3!}\Delta^3 y_0 + \frac{(4p^3-18p^2+22p-6)}{4!}\Delta^4 y_0 + ...\right]$$

The difference table is given by

x	$p = \frac{x-50}{10}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	0	**19.96**				
			16.69			
60	1	36.65		**5.47**		
			22.16		**-9.23**	
70	2	58.81		-3.76		**11.99**
			18.40		2.76	
80	3	77.21		-1.00		
			17.40			
90	4	94.61				

$$\therefore\left(\frac{dy}{dx}\right)_{p=0.1}=\frac{1}{10}\left[16.69+\frac{(0.2-1)}{2}(5.47)+\left[\frac{3(0.1)^2-6(0.1)+2}{6}\right](-9.23)+\frac{\left[4(0.1)^3-18(0.1)^2+22(0.1)-6\right]}{24}\times 11.99+\ldots\right]$$

$$=\frac{1}{10}[16.69-2.188-2.1998-1.9863]=1.0316$$

$$\left(\frac{d^2y}{dx^2}\right)_{p=0.1}=\frac{1}{h^2}\left[\Delta^2 y_0+(p-1)\Delta^3 y_0+\frac{(6p^2-18p+11)}{12}\Delta^4 y_0+\ldots\right]$$

$$=\frac{1}{100}\left[5.47+(0.1)-1(-9-23)+\frac{\left[6(.1)^2-18(.1)+11\right]}{12}\right]\times 11.99$$

$$=\frac{1}{100}[5.47+8.307+9.2523]$$

=0.2303.

Problem 2. The population of a certain town is shown in the following table

Year x	1931	1941	1951	1961	1971
Population y	40.62	60.80	79.95	103.56	132.65

Find the rate of growth of the population in 1961.

Solution. Here h = 10 Since the rate of growth of population is $\frac{dy}{dx}$ we have to find $\frac{dy}{dx}$ at x = 1961, which lies nearer to the end value of the table. Hence we choose the origin at x = 1971 and we use Newton's backward interpolation formula for derivative.

$$\frac{dy}{dx}=\frac{1}{h}\left[\nabla y_4+\frac{(2p+1)}{2}\nabla^2 y_4+\frac{(3p^2+6p+2)}{6}\nabla^3 y_4+\frac{(2p^3+9p^2+11p+3)}{12}\nabla^4 y_4+\ldots\right]$$

Where $p=\frac{x-x_0}{10}=\frac{1961-1971}{10}=-1$

The backward difference table

x Year	y Population	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62				
		20.18			
1941	60.80		-1.03		
		19.15		5.49	**-4.47**
1951	79.95		4.46		
		23.61		**1.02**	
1961	103.56		**5.48**		
		29.09			
1971	132.65				

$$\left(\frac{dy}{dx}\right)_{p=-1} = \frac{1}{10}\left[29.09 + -\left(\frac{1}{2}\right)(5.48) + \frac{\left[3(-1)^2 + 6(-1) + 2\right]}{6} \times 1.02 + \frac{\left[2(-1)^3 + 9(-1)^2 + 11(-1) + 3\right]}{12}(-4.47)\right]$$

$$= \frac{1}{10}\left[29.09 - 2.74 - 0.17 + 0.3725\right]$$

$$= \frac{1}{10}\left[26.5525\right] = 2.6553$$

$\therefore$ The rate of growth of the population in the year 1961 is 2.6553.

Numerical Integration

Given set of (n + 1) data points (x_i, y_i), i = 0,1,2,.......,n of the function y = f(x), where f(x) is not known explicitly, it is required to evaluate $\int_{x_0}^{x_n} f(x)\,dx$.

Newton-Cote's Quadrature Formula (General Quadrature Formula):

This is the most popular and widely used numerical integration formula. It forms the basis for a number of numerical integration methods known as Newton-Cote's methods.

Derivation of Newton-Cotes formula:

Let the interval [a, b] be divided into n equal sub-intervals such that $a = x_0 < x_1 < x_2 < x_3 \ldots\ldots\ldots < x_n = b$. Then $x_n = x_0 + nh$.

Newton forward difference formula is

$$y(x) = y(x_0 + ph) = P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots \quad (1)$$

Where $p = \frac{x \quad x_0}{h}$.Now, instead of f(x) we will replace it by this interpolating polynomial.

$$\therefore \int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_n} P_n(x)\,dx$$, where $P_n(x)$ is an interpolating polynomial of degree n

$$= \int_{x_0}^{x_0+nh} P_n(x)\,dx = \int_{x_0}^{x_0+nh}\left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots \right]dx$$

Since $x = x_0 + ph$, $dx = h.dp$ and hence the above integral becomes

$$\int_{x_0}^{x_n} f(x)\,dx = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots \right]dp$$

$$= h\left[y_0(p) + \frac{p^2\Delta y_0}{2} + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - 3\frac{p^3}{3} + 2\frac{p^2}{2}\right)\Delta^3 y_0 + \ldots\ldots \right]$$

$$= h\left[ny_0 + \frac{n^2\Delta y_0}{2} + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - 3\frac{n^3}{3} + 2\frac{n^2}{2}\right)\Delta^3 y_0 + \ldots\ldots \right]$$

$$= nh\left[y_0 + \frac{n\Delta y_0}{2} + \frac{1}{2}\left(\frac{n^2}{3} - \frac{n}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^3}{4} - 3\frac{n^2}{3} + 2\frac{n}{2}\right)\Delta^3 y_0 + \ldots\ldots \right]$$

This is called **Newton-Cote's Quadrature for)mula.** ....(2)

Trapezoidal Rule:

Putting n = 1 in the above general formula, all differences higher than the first will become zero (since other differences do not exist if n = 1) and we get

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx = \text{h}\left[y_0 + \frac{1}{2}\Delta y_0\right] = \text{h}\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1)$$

and $$\int_{x_1}^{x_2} f(x)\,dx = \int_{x_0+h}^{x_0+2h} f(x)\,dx = \text{h}\left[y_1 + \frac{1}{2}\Delta y_1\right] = \text{h}\left[y_1 + \frac{1}{2}(y_2 - y_1)\right] = \frac{h}{2}(y_1 + y_2)$$

$$\int_{x_2}^{x_3} f(x)\,dx = \int_{x_0+2h}^{x_0+3h} f(x)\,dx = \text{h}\left[y_2 + \frac{1}{2}\Delta y_2\right] = \text{h}\left[y_2 + \frac{1}{2}(y_3 - y_2)\right] = \frac{h}{2}(y_2 + y_3)$$

....

Finally,

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Hence,

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+3h} f(x)\,dx + \ldots.. + \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{h}{2}[y_0 + y_1] + \frac{h}{2}[y_1 + y_2] + \ldots\ldots\ldots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}\left[(y_0 + y_1) - 2(y_1 + y_2 + y_3 + y_4 + \ldots\ldots + y_{n-2} + y_{n-1}\right]$$

.... (3)

Simpson's 1/3 Rule

Putting n = 2 in Newton-Cotes Quadrature formula i.e., by replacing the curve y = f(x) by n/2 parabolas, we have

$$\int_{x_0}^{x_2} f(x)\,dx = 2h\left[y_0 + \frac{2}{2}\Delta y_0 + \frac{2(4-3)}{12}\Delta^2 y_0\right] = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right] \quad = 2h\left[\frac{1}{6}y_0 + \frac{2}{3}y_1 + \frac{1}{6}y_2\right]$$

$$= \frac{2h}{6}\left[y_0 + 4y_1 + y_2\right] = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly, $\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$

....

$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$ Adding all these integrals, we obtain

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots\dots\dots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] + \frac{h}{3}\left[y_2 + 4y_3 + y_4\right] + \dots\dots\dots + \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

$$= \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \ldots\ldots\ldots\ldots + (y_{n-2} + 4y_{n-1} + y_n)]$$

$$= \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_2 + y_3 + y_5 + y_{n-1}) + 2(y_2 + y_4 + y_6 + \ldots\ldots + y_{n-2})]$$

………………… (4)

$$= \frac{h}{3}\left[\begin{array}{l} sum\ of\ the\ first\ and\ last\ ordnates) + 4(sum\ of\ the\ odd\ ordinates) \\ + 2(sum of\ the\ remaining\ even\ ordinates \end{array}\right]$$

With the convention that y_0, y_2, y_4,….., y_{2n} are even ordinates and y_1, y_3, y_5,….., y_{2n-1} are odd ordinates.

This is known as **Simpson's 1/3 rule** or simply **Simpson's rule**.

Simpson's 3/8 Rule:

n = 3 in Newton-Cote's Quadrature formula, all differences higher than the third will become zero and we obtain

$$\int_{x_0}^{x_3} f(x)\,dx = 3\text{h}\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3(6-3)}{12}\Delta^2 y_0 + \frac{3(3-2)^2}{24}\Delta^3 y_0 \right]$$

$$\int_{x_0}^{x_3} f(x)\,dx = 3\text{h}\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0 \right]$$

$$\int_{x_0}^{x_3} f(x)\,dx = 3\text{h}\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0) \right]$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3}{8}\text{h}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly,

$$\int_{x_3}^{x_6} f(x)\,dx = \frac{3}{8}h\left[y_3 + 3y_4 + 3y_5 + y_6\right] \quad \text{and so on.}$$

Adding all these integrals, from x_0 to x_n, where n is a multiple of 3, we get

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \dots\dots\dots + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

$$= \frac{3h}{8}\left[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \dots\dots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\right]$$

$$= \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots\dots\dots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \dots\dots + y_n)\right]$$

.... (5)

Equation (5) is called **Simpson's 3/8 rule** which is applicable only when n is multiple of 3.

Problems : Evaluate $\int_0^1 \frac{dx}{1+x}$ using (i) Trapezoidal rule (ii) Simpson's one third rule (iii) Simpson's three eight rule. Take $h = \frac{1}{6}$ for all cases.

Solutions: Here $h = \frac{1}{6}$, Let $y = f(x) = \frac{1}{1+x}$. The values of f(x) for the points of subdivisions are as follows:

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = \frac{1}{1+x}$	1	0.8571	0.75	0.6667	0.6	0.5455	0.5

(i) Tapezoidal rule

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

□ $\frac{1}{12}\left[\left(1+0..5\right)+2\left(0.8571+0.755+0.6667+0.6+0.5455\right)\right]$

=0.6949.

(ii) Simpson's one third rule

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}\left[\left(y_0+y_6\right)+2\left(y_2+y_4\right)+4(y_1+y_3+y_5)\right]$$

□ $\frac{1}{18}\left[\left(1+0.5\right)+2\left(0.75+0.6\right)+4(0.8571+0.6667+0.5455)\right]$

= 0.6932.

(iii) Simpson's three eight rule

$$\int_0^1 \frac{dx}{1+x} = \frac{3h}{8}\left[\left(y_0+y_6\right)+3\left(y_1+y_2+y_4+y_5\right)+2y_3\right]$$

□ $\frac{1}{16}\left[\left(1+0.5\right)+3\left(0.8571+0.75+0.6+0.5455+2\left(0.6667\right)\right)\right]$

= 0.6932.

Assignment-Cum-Tutorial Questions
UNIT-III
SECTION-A

Objective Questions

1. By Newton's forward interpolation formula

$\frac{dy}{dx}=$ ______________________________

$\frac{d^2y}{dx^2}=$ ______________________________

2. By Newton's backward interpolation formula

$\frac{dy}{dx}=$ ______________________________

$\frac{d^2y}{dx^2}=$ ______________________________

3. In the second derivative using Newton's backward difference formula, the coefficient of $\nabla^2 f(a)$

(a) $-1/h^2$ (b) $1/h^2$ (c) 11/12 (d) $-h^2$

4. Trapezoidal rule to find definite integral is____________________

5. Simpson's $1/3^{rd}$ rule to find definite integral is____________________

6. Simpson's $3/8^{th}$ rule to find definite integral is____________________

7. If we put n = 2 in a general quadrature formula, we get [ ]

(a) Trapezoidal rule (b) Simpson's $1/3^{rd}$ rule

(c) Simpson's $3/8^{th}$ rule (d) Boole's rule

8. In Simpson's $1/3^{rd}$ rule the number of subintervals should be [ ]

(a) Even (b) odd

(c) multiples of 3's (d) more than 'n' interval

9. If the distance *d(t)* is traversed by a particle in the 't' sec and *d(0) = 0, d(2) = 8, d(4) = 20 and d(6) = 28*, then its velocity in cm after 6 secs is [ ]

(a) 1.67 (b) 16.67 (c) 2 (d) 2.003

10. The formula $\frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + ...\right]$ is used only when the point x is at [ ]

(a) end of the tabulated set (b) middle of the tabulated set

(c) Beginning of tabulated set (d) none of these

11. To increase the accuracy in evaluating a definite integral by Trapezoidal rule, we should take ______________________

12. Values of y = f(x) are known as x = x_0, x_1 and x_2. Using Newton's forward integration formula, the approximate value of $\left(\frac{dy}{dx}\right)_{x=x_0}$ is ______________

13. Numerical differentiation gives [ ]

(a) exact value (b) approximate value

(c) no result (d) negative value

14. For n =1in quadrature formula, $\int_{x_0}^{x_1} f(x)\,dx$ equals to [ ]

(a) $\frac{h}{2}(f_0 + f_1)$ (b) $(f_0 + f_1)$ (c) $\frac{h}{2}(f_0 - f_1)$ (d) $\frac{h}{4}(f_0 + f_1)$

15. To apply, Simpson's 1/3rd rule, always divide the given range of integration into 'n' subintervals, where n is [ ]

(a) even (b) odd (c) 1,2,3,4 (d) 1,3,5,7

16. The process of calculating derivative of a function at some particular value of the independent variable by means of a set of given values of that function is [ ]

(a) Numerical value (b) Numerical differentiation

(c) Numerical integration (d) quadrature

17. While evaluating definite integral by Trapezoidal rule, the accuracy can be increased by [ ]

(a) h =4 (b) even number of sub-intervals

(c) multiples of 3 (d) large number of sub-intervals

Section-B

Subjective Questions

1. A curve is expressed by the following values of x and y. Find the slope at the point x =1.5

X	0.0	0.5	1.0	1.5	2.0
Y	0.4	0.35	0.24	0.13	0.05

2. The population of a certain town is given below. Find the rate of growth of the population in 1961:

Year	1931	1941	1951	1961	1971
Population	40.62	60.80	71.95	103.56	132.65

3. In a machine a slider moves along a fixed straight rod. Its distance x cms along the rod is given below for various values of time 't' seconds. Find the velocity and acceleration of the slider when t = 0.3

t(sec)	0	0.1	0.2	0.3	0.4	0.5	0.6
x(cms)	30.13	31.62	32.87	33.64	33.95	33.81	33.24

4. The velocity of a train which starts from rest is given by the following table being reckoned in minutes from the start and speed in miles per hour

Minutes	2	4	6	8	10	12	14	16	18
Miles per hour	10	18	25	29	32	20	11	5	2

Estimate approximately the total distance travelled in 20 minutes.

5. The distance covered by an athlete for the 50 meter is given in the following table

Time(sec)	0	1	2	3	4	5	6
Distance(meter)	0	2.5	8.5	15.5	24.5	36.5	50

Determine the speed of the athlete at t = 5 sec. correct to two decimals.

6. A curve is drawn to pass through the points given by following table:

X	1	1.5	2.0	2.5	3	3.5	4.0
Y	2	2.4	2.7	2.8	3	2.6	2.1

Find the slope of the curve at x=1.25.

7. Evaluate $\int_0^2 e^{-x^3} dx$ using Simpson's rule taking h=0.25

8. A river is 80 meters wide. The depth 'd' in meters at a distance x from the bank is given in the following table. Calculate the cross section of the river using Trapizoidal rule.

x	10	20	30	40	50	60	70	80
d(x)	4	7	9	12	15	14	8	3

9. Compute the value of the definite integral $\int_4^{5.2} \log x dx \; or \int_4^{5.2} \ln x dx$ using i.Trapezoidal Rule ii. Simpson's 1/3rd Rule and iii. Simpson's 3/8th Rule.

10. The following table gives the velocity v of a particle at time 't'

t (seconds)	0	2	4	6	8	10	12
v meters per second	4	6	16	34	60	94	136

Find (i)the distance moved by the particle in 12 seconds and also (ii) the acceleration at t = 2 sec

11. Using Simpson's 1/3rd rule, find the value of the integral $\int_{0.2}^{1.4} (\sin x - \log x + e^x) dx$ by taking 6 sub-intervals.

Section-C

GATE/IES/Placement Tests/Other competitive examinations

1. If $f(2) = 5, f(4) = 8, f(6) = 10,$ *and* $f(8) = 16$ then $f''(8)$ = __________

2. Using Simpson's 1/3rd rule, find the value of the integr $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ by taking 6 sub-intervals.

3. Minimum number of subintervals required to evaluate the integral $\int_{1}^{2} \frac{1}{x} dx$ by using Simpson's 1/3rd rule so that the value is corrected up to 4 decimal places.

4. The following table gives the velocity v of a particle at time 't'

t (seconds)	0	2	4	6	8	10	12
v meters per second	4	6	16	34	60	94	136

Find (i) the distance moved by the particle in 12 seconds and also (ii) the acceleration at t = 2 sec.

5. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's 1/3rd rule, find the velocity of the rocket at t= 80 seconds.

t sec	0	10	20	30	40	50	60	70	80
f(cm/sec^2)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

Unit –IV

FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

Objectives:

To introduce essential methods to solve 1st order ODE and applications of 1st order ODE such as Newton's law of cooling and orthogonal trajectories.

Syllabus:

Exact and non-exact D.E., Applications : Newton's law of cooling and orthogonal trajectories.

Outcomes:

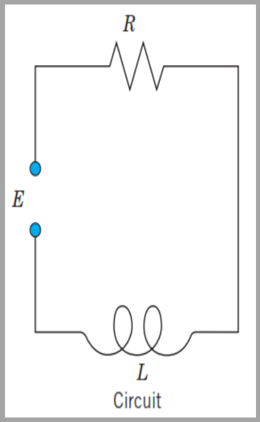
At the end of the unit Students will be able to

- differentiate exact and non-exact D.E
- solve exact and non-exact D.E
- apply the concept of Newton's law of cooling
- find orthogonal trajectory of given family of curves.

***INTRODUCTION* :**

- If we want to solve an engineering problem (usually of a physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions and equations. Such an expression is known as a ***mathematical model*** of the given problem.
- The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called mathematical modeling or, briefly, ***modeling.***
- Many physical concepts, such as velocity and acceleration, are derivatives. Hence a model is very often an equation containing derivatives of an unknown function. Such a model is called a ***differential equation.***
- Hence any Physical situation involving motion or measure rates of change can be described by a mathematical model, the model is just a differential equation.

Formation of Differential equations for real life problems →



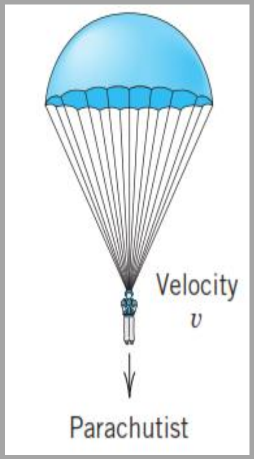
Modeling RL-Circuit :
In this case, we use the following Physical Laws to create mathematical model.

[Ohm's law] → A current I in the circuit causes a **voltage drop RI** across the resistor

[Kirchoff's Voltage law] → A voltage drop $L\frac{dI}{dt}$ across the conductor, and the sum of these two voltage drops equals the EMF.

According to the above laws, the differential equation corresponding to the model is given by

$$L\frac{dI}{dt} + RI = E(t)$$



Parachutist. Two forces act on a parachutist, the attraction by the earth mg (m = mass of person plus equipment, $g = 9.8 \text{ m/sec}^2$ the acceleration of gravity) and the air resistance, assumed to be proportional to the square of the velocity $v(t)$. Using **Newton's second law** of motion (mass × acceleration = resultant of the forces),

- Under the assumption that the force of air resistance is proportional to velocity and opposes the motion, the second-order equation of motion for a parachutist falling in a coordinate system where x(t) is measured positive upward from the ground, is

$$m\frac{d^2}{dt^2}x(t) = -m.g - k.\left[\frac{d}{dt}x(t)\right]$$. Where x(t) → displacement.

Differentiation :

- The rate of change of a variable w.r.t the other variable is called a differentiation.

In this case, changing variable is called *Dependent variable* and other variable is called an *Independent variable.*

Example : $\frac{dy}{dx}$ *is known as differentiation where y is dependent variable and x is independent variable.*

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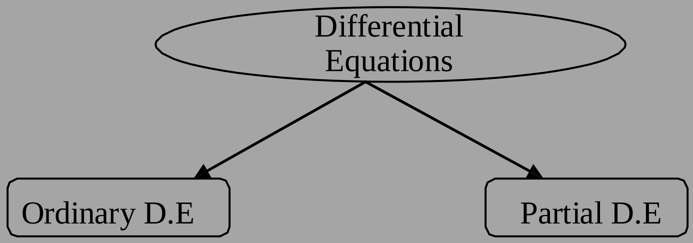
Differential Equations are separated into two types



- Ordinary D.E: In a D.E if there exist ***single Independent variable***, it is called as Ordinary D.E.

Example: 1) $\frac{dy}{dx}+2y=0$ is an Ordinary D.E 2) $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+1=0$ is an Ordinary D.E.

- Partial D.E: In a D.E if there exist ***more than one Independent variables*** then it is called as Partial D.E

Example: 1) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ is a Partial D.E.Here u depends on two independent variables x & y.

2) $\frac{\partial^2 u}{\partial x \partial y}+1=0$ is a Partial D.E. Here u depends on two independent variables x & y.

****

- **Order of D.E. :**

The order of the D.E. is the order of the highest derivative involving in the equation.

Example : **1)** Order of $\frac{d^2y}{dx^2}+2y=0$ is **Two.** **2)** Order of $\frac{d^5y}{dx^5}+\left[\frac{d^3y}{dx^3}\right]^8+3y=0$ is **Five**

- **Degree of D.E.:**

The degree of the D.E is the degree of the highest ordered derivative involving in the equation, when the equation is free from radicals and fractional terms.

Example: 1) The degree of $\left[\frac{d^2y}{dx^2}\right]^1+2\frac{dy}{dx}+1=0$ is One.

2) The degree of $x\left[\frac{d^2y}{dx^2}\right]^8+\left[\frac{dy}{dx}\right]^{11}+\left[\frac{d^3y}{dx^3}\right]^2=0$ is Two.

ODE :

- Ordinary differential equation is an equation involving dependent variable (y) and its derivatives (y^1, y^{11},…) with respect to the independent variable (x).

Examples : $\frac{dy}{dx}+xy^2-4x^3=0$, $\frac{d^2y}{dx^2}+4\frac{dy}{dx}-3y=x^2-7$, …..

****

1st Order ODE :

- 1st Order Ordinary differential equation is an equation involving dependent variable (y) and its derivative y^1 with respect to the independent variable (x).

Examples : $\frac{dy}{dx}+xy^2-4x^3=0$

Solving 1st order & 1st degree ODE

We are going to solve the 1st order ODEs by the following methods.

1. Exact DE   2. Non-exact DE

Exact DE

- **Definition :** A D.E. which can be obtained by direct differentiation of some function of x and y is known as exact differential equation.

- Necessary & Sufficient condition for the D.E. of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ to be ***Exact*** *is* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

- **Procedure to solve Exact D.E.:**

Step 1 : Identify M and N

Step 2: Check of Exactness.

Step 3: If exact, General Solution is

$\int M\ dx + \int N\ dy = C$ **[In N, take terms which do not have x variable**

While integrating M take y as constant]

Problems :

1. Solve

$$\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\}dx + (x + \log x - x\sin y)\,dy = 0$$

Solution :

Step 1 : $Clearly\ \ M = y + \frac{y}{x} + Cos\,y$ & $N = x + \log x - x.Sin\,y$

$$\therefore \frac{\partial M}{\partial y} = \left(1+\frac{1}{x}\right) - \sin y \text{ and } \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

Step 2: $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence the given equation is exact

No "x" terms

Step 3: General Solution is given by $\int M\ dx + \int N\ dy = C$

$$\int \left\{ y\left(1+\frac{1}{x}\right) + \cos y \right\} dx + \int 0\,dy = C \Rightarrow y.(x+\log x) - Cosy.(x) = c$$

---

2. Solve (1 – sin x tan y) dx + (cos x sec^2 y) dy = 0

Solution :

Step 1 : Clearly , M = 1 – sin x tan y and N = cos x sec^2 y

Step 2 : $\frac{\partial M}{\partial y} = -\sin x \sec^2 y = \frac{\partial N}{\partial x}$ $\Rightarrow$ Exact Differential

Step 3 : Hence General solution :

$$\int M\ dx + \int N\ dy = C$$

$$\Rightarrow \int (1 - Sinx.Tany)dx + 0dy = c$$

$$\Rightarrow x - (Tany).(-Cosx) = c \quad \text{or} \quad x + (Tany).(Cosx) = c$$

****

Non-Exact DE

- **If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the D.E. M(x, y) dx + N(x, y) dy = 0 is said to be Non-Exact Differential equation.**

- **Step 1 : Identify M and N**

  Step 2: Check of Exactness.

  Step 3: If Non- Exact, <u>Convert the given D.E. to EXACT D.E</u> Using the Integrating Factor by the following suitable method.

METHOD – 1 : Method to find Integrating factor $\frac{1}{Mx+Ny}$

If given D.E. Mdx + Ndy = 0 is Non-Exact and ***<u>M, N are homogeneous functions of same degree</u>***, then I.F. = $\frac{1}{Mx+Ny}$ $(Mx+Ny \neq 0)$

METHOD –2 : Method to find Integrating factor $\frac{1}{Mx - Ny}$

If given D.E. Mdx + Ndy = 0 is Non-Exact and ***M is of the form y.f(xy) &***
N is of the form x.g(xy), *then* $I.F. = \frac{1}{Mx - Ny} \quad (Mx - Ny \neq 0)$

METHOD –3 : Method to find Integrating factor $e^{\int f(x)dx}$

If given D.E. Mdx + Ndy = 0 is Non-Exact and

if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ = a function of x alone =f(x) then I.F. = $e^{\int f(x)dx}$

METHOD –4 : Method to find Integrating factor $e^{\int g(y)dy}$

If given D.E. Mdx + Ndy = 0 is Non-Exact and

if $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ = a function of y alone =g(y) then I.F. = $e^{\int g(y)dy}$

METHOD –5 : [Inspection Method] Observe the D.E. and if possible split the D.E. into any of the following R.H.S. and Integrate.

a. $d\left(\frac{x^2+y^2}{2}\right) = xdx + ydy$

b. $d(xy) = xdy + ydx$

c. $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

d. $d\left(\frac{e^y}{x}\right) = \frac{x.e^y dy - e^y dx}{x^2}$

e. $d\left(\log\frac{y}{x}\right) = \frac{xdy - ydx}{xy}$

$d\left(\log\frac{x}{y}\right) = \frac{ydx - xdy}{xy}$

g. $d\left(Tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2+y^2}$

h. $d\left(Tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

Example 1 :

1. Solve $x^2y\,dx - (x^3 + y^3)\,dy = 0$

Solution :

Step 1 : Here $M = x^2y$ and $N = -x^3 - y^3$

Clearly, $\frac{\partial M}{\partial y} = -3y^2 \neq \frac{\partial N}{\partial x} = 2xy$.

∴ Non-Exact D.E

Step 2 : As M & N are homogeneous functions of same degree 3,

[Method 1 follows]

I.F. = $\frac{1}{Mx + Ny} = -\frac{1}{y^4} \neq 0$

Step 3 : Multiply the Give D.E. With the I.F.,

$$-\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

Step 4 : Clearly $M = \frac{-x^2}{y^3}$ and $N = \frac{x^3}{y^4} + \frac{1}{y}$

Step 5 : observe that $\frac{\partial M}{\partial y} = \frac{3x^2}{y^4}$ and $\frac{\partial N}{\partial x} = \frac{3x^2}{y^4}$

∴ Exact D.E

Step 6 : General Solution becomes → $\int M\ dx + \int \underset{\smile}{N}\ dy = C$

No "x" terms

Step 7 : $\int -\frac{x^2}{y^3}dx + \int \frac{1}{y}dy = C \quad \Rightarrow \left(-\frac{1}{y^3}\right).\frac{x^3}{3} + \log y = C$

Note : similar steps are applicable for the non-exact D.E.s which will come under Methods 2,3 and 4.

Example 2 : [Method 5] Solve $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$:

Note : (We can also use Method 2)

Solution : Given equation can be written as $(y\,dx + x\,dy) + (xy^2\,dx - x^2y\,dy) = 0$

Or $d\,(yx) + xy^2\,dx - x^2\,y\,dy = 0$

Dividing by x^2y^2, $\frac{d(xy)}{x^2y^2} + \frac{1}{x}dx - \frac{1}{y}dy = 0$

Integrating, $$\int \frac{d(xy)}{(xy)^2} + \int \frac{1}{x}dx - \int \frac{1}{y}dy = C$$

$$\Rightarrow \frac{(xy)^{-1}}{-1} + \log x - \log y = c$$

$$\Rightarrow \frac{(xy)^{-1}}{-1} + \log x - \log y = c$$

Applications of 1st order ODE

Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming). If T is the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's law of cooling as a differential equation :

$$\frac{dT}{dt} = -k(T - T_s)$$ solving , $$\boxed{T - T_s = c.e^{-kt}}$$ where k> 0

1. A cup of tea at temperature 90^0 C is placed in a room having temperature 25^0C. It cools to 60^0C in 5 minutes. Find the temperature after an interval of 5 minutes.

Solution : The problem can be classified as →

Stage 1 : T=90^0C → t=0		: C value
Stage 2 : T=60^0C → t=5 Mins		: k value
Stage 3 : T= ? → t = 10 Mins.		

This problem will come under ***Newton's law of cooling.***

Here $T_s = 25^0$C

We use the solution $\boxed{T - T_s = c.e^{-kt}}$ to solve the above problem.

Step 1 : **C** : T = 90, t = 0 $\Rightarrow$ $90 - 25 = Ce^{-k0}$ $\Rightarrow$ **C = 65.**

Step 2 : **k** : T= 60 → t = 5 Mins. $\Rightarrow$ $60 - 25 = 65.e^{-k5}$ $\Rightarrow$ $e^{-5k} = 0.53846$

$\Rightarrow$ - 5 k = ln (0.53846) $\Rightarrow$ **k** = 0.619/5 = **0.1238**

Step 3 : T : When t = 10 Mins, $T - 25 = 65.e^{-(0.1238)10}$ $\Rightarrow$ $T = 25 + 65.e^{-1.238}$

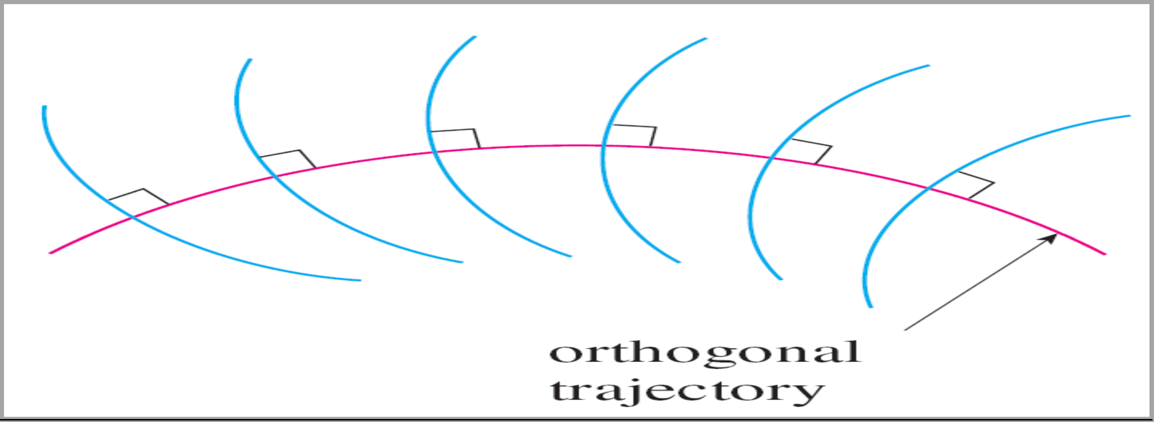
$\Rightarrow$ T = 25 + 65(0.2899)

$\Rightarrow$ $T = 44^0$ (Appx.)

Hence in 10 Mins. the temperature of the tea would be 44^0

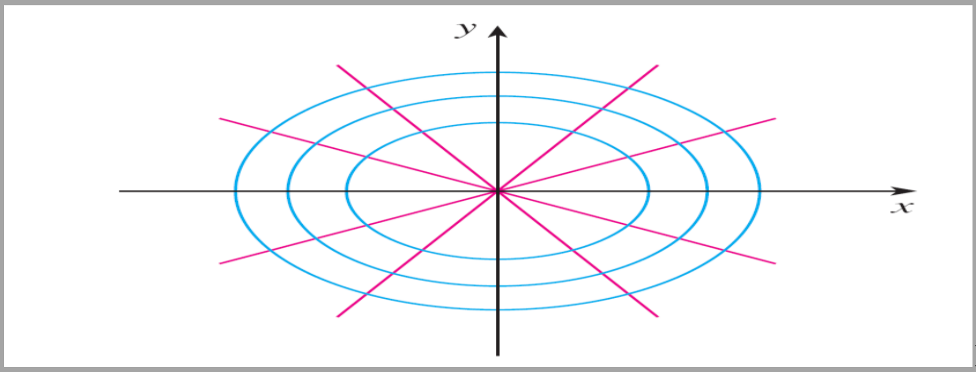
Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles



For example, each member of the family y = mx of straight lines through the origin is an orthogonal trajectory of thefamily $x^2 + y^2 = r^2$ of concentric circles with center the origin .

We say that the two families are orthogonal trajectories of each other.



NOTE: Orthogonal trajectories has important applications in field of physics . equipotential lines and the streamlines in an irrotational 2D flow are orthogonal. In an electrostatic field,the lines of force are orthogonal to the lines of constant potential.The streamlines in aerodynamics are orthogonal trajectories of the velocity-equipotential curves.

A procedure for finding a family of orthogonal trajectories F(x, y, C) = 0

for a given family of curves F(x, y, C) = 0 is as follows:

Step 1: Determine the differential equation for the given family F(x, y, C) = 0.

Step 2: Replace $y^{|}$ in that equation by $-1/y^{|}$; the resulting equation is the differential equation for the family of orthogonal trajectories.

step 3: Find the general solution of the new differential equation. This is the family of orthogonal trajectories.

Example : Find the orthogonal trajectories of the family of curves $x = ky^2$, where is k an arbitrary constant.

Solution:The curves $x = ky^2$ form a family of parabolas whose axis of symmetry is the x-axis.The first step is to find a single differential equation that is satisfied by all members of the family.If we differentiate $x = ky^2$, we get

$$1 = 2ky\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{2ky}$$

This differential equation depends on *k*, but we need an equation that is valid for all values of *k* simultaneously.

To eliminate *k* we note that, from the equation of the given general parabola $x = ky^2$, we have $k = x/y^2$ and so the differential equation can be written as

$$\frac{dy}{dx} = \frac{y}{2x}$$

Or This means that the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = y/(2x)$.

On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope.Therefore the orthogonal trajectories must satisfy the differential equationThis differential equation is separable, and we solve it as follows:

$$\frac{y^2}{2} = -x^2 + C$$

$$x^2 + \frac{y^2}{2} = C$$

Note: In polar coordinates after getting the differential equation of the family of curves,we have to replace **dr/d*θ* by -r² dθ/dr** and then integrate the resulting differential equation

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A. ***Objective Questions***

1. The solution of $\frac{dy}{dx} = e^{x+y}$ is [ ]

   a) $e^{-x} + e^{-y} = c$ b) $e^{x} + e^{-y} = c$ c) $e^{-x} + e^{y} = c$ d) $e^{x} + e^{y} = c$

2. Integrating factor of $\frac{dy}{dx} + \frac{y}{x} = \frac{\log x}{x}$ is [ ]

   a) logx b) x c) $\frac{1}{x}$ d) e^{x}

3. For the differential equation $(y + 3x)\,dx + x\,dy = 0$, the particular solution when $x = 1$, $y = 3$ is [ ]

   a) $3y^2 + 2xy = 9$ b) $3x^2 + 2y^2 = 21$ c) $3x^2 + 2y = 9$ d) $3x^2 + 2xy = 9$

4. Orthogonal trajectories of $r = ce^{\theta}$ is [ ]

   a) $r = k\log(\theta)$ b) $r\log\theta = k$ c) $r = k\,e^{-\theta}$ d) $r\,e^{-\theta} = k$

5. The equation of family of curves that is orthogonal to the family of curves represented by $r\theta = c$ is givenby [ ]

a) $r = ae^{\theta}$ b) $r = ae^{-\theta}$ c) $r = a^{\theta}$ d) $r = a^2 e^{\theta^2/2}$

6. Orthogonal trajectory of the curves A= $r^2 cos\theta$ are [ ]

a)A=r sin θ b) B= r^2 sin θ c) B= r cosθ d) B= r^2 cos θ

7. The solution to the exact D.E. $(x^2 - y^2 + 1)\,dx + (1 - 2xy)dy = 0$ is [ ]

a) $\frac{x^3}{3} - xy^2 + x + y = c$ b) $\frac{x^3}{3} - xy^2 - x - y = c$

c) $\frac{x^3}{3} - xy^2 + x = c$ d) $\frac{x^3}{3} - xy^2 - x = c$

8 Mdx + Ndy = 0 is exact if [ ]

a) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$ b) $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$ c) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ d) $\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = 0$

9. Find the integrating factor to convert non-exact D.E. $2xy\,dy - (x^2+y^2+1)dx=0$ to exact D.E. [ ]

a) y^2 b) x^2 c) $\frac{1}{y^2}$ d) $\frac{1}{x^2}$

10. Find the integrating factor to convert non-exact D.E. (y.logy) dx + (x – logy) dy = 0 to exact D.E. [ ]

a) y b) –y c) $-\frac{1}{y}$ d) $\frac{1}{y}$

11.The equation of the family of orthogonal trajectories of the system of parabolas $y^2 = 2x + C$ is [ ]

a) $y = Ce^{-x}$ b)$y = Ce^{2x}$ c)$y = Ce^{x}$ d) $y = Ce^{-2x}$

12.Which of the following equations is an exact D.E.? [ ]

a) $(x^2 + 1)\,dx - xy\,dy = 0$ b) $x\,dy + (3x - 2y)\,dx = 0$
c) $2xy\,dx + (2 + x^2)\,dy = 0$ d) $x^2y\,dy - y\,dx = 0$

13) Degree of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/3} = c.\frac{d^2y}{dx^2}$ is ______________

14) Order of the differential equation $\left[\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]^{1/3}=\frac{d^2y}{dx^2}$ is ______________

15) The solution of $\cos y\frac{dy}{dx}+\sin y=e^{-x}$ is ______

16) Solution of a differential equation which is not obtained from the general solution is known as ______________________

17) Solution of $(x + 1)dy + (y + 2)dx = 0$ is ______________________

18) The integrating factor of $M\,dx + N\,dy = 0$, where M & N are homogeneous functions of same degree, is ____________________

19) The integrating factor of $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$ is ___________________

20) $(xdy + ydx) = d($________$)$

B.Subjective Questions:

1) Solve $(x^2-ay)dx=(ax-y^2)dy$
2) Solve $\left(1+e^{x/y}\right)dx+\left(1-x/y\right)e^{x/y}dy=0$
3)
4) Solve: $xdx+ydy=\frac{xdy-ydx}{x^2+y^2}$
5) Solve : $\frac{dy}{dx}-\frac{Tan\,y}{1+x}=(1+x)e^x\sec y$.
6) If the temperature of a body changes from 100°C to 70°C in 15 minutes, find when the temperature will be 40°C, if the temperature of air is 30° C.
7) The temperature of the body drops from 100^0C to 75^0C in ten minutes. When the surrounding air is at 20^0C temperature. What will be its temperature after half an hour? When will the temperature be 25^0?
8) The number of N of bacteria in a culture grew at a rate Proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after 1 ½ hours?
9) Show that the system of confocal conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ is self Orthogonal.
10) Find orthogonal trajectories of $r^n\sin n\theta=a^n$
11) Find orthogonal trajectory of $r=a(1+\cos\theta)$

C.GATE QUESTIONS

1. A body originally at 60^{O}C cools down to 40^{O}C in 15 minutes when kept in air at a temperature of 25^{O}C. What will be the temperature of the body at the end of 30 minutes? [GATE – 2007] [ ]

(a) 35.2^{O} C (b) 31.5^{O} C (c) 28.7^{O} C (d) 15^{O} C

2. Solution of the differential equation 3y dy/dx +2x = 0 represents a family of [GATE – 2009] [ ]

(a)Ellipses (b) circles (c) Parabolas (d) hyperbolas

3. Match each differential equation in Group I to its family of solution curves from Group II. [ ]
[GATE-2009]

Group I	Group II
P. dy / dx = y/x.	1. Circles
Q. dy / dx = −y/x	2. Straight lines
R. dy / dx = x/y.	3. Hyperbolas
S. dy / dx = −x/y	

Codes:

	P	Q	R	S		P	Q	R	S
(a)	2	3	3	1	(b)	1	3	2	1
(c)	2	1	3	3	(d)	3	2	1	2

4. A D.E of the form dy/dx=f(x, y) is homogeneous if the function f(x, y) depends only on the ratio y/x or x/y [GATE:1995] [TRUE / FALSE]

5. The solution of $\frac{dy}{dx} + y^2 = 0$ is [GATE:2003] [ ]

a)$y = \frac{1}{x+c}$ b)y=$\frac{-x^3}{3}$+c c)y=ce^x d)unsolvable as equation is nonlinear

6. The solution of $\frac{dy}{dx} = y^2$ with initial value y(0)=1 bounded in the interval [GATE:2007] [ ]

a)$-\infty \le x \le \infty$ b) $-\infty \le x \le 1$ c)x<1,x>1 d) -2$\le x \le 2$

7. For the D.E. $\frac{dy}{dx} + 5y = 0$ with y(0)=1 the general solution is [GATE:1994] [ ]

a)e^{5t} b) e^{-5t} c)5 e^{-5t} d) $e^{\sqrt{-5t}}$

8. Which of the following is a solution to the D.E.$\frac{dx(t)}{dt} + 3x(t) = 0$? [GATE:2008] [ ]

a) x(t)=3e^{-1} b) x(t)= $2\,e^{-3t}$ c) x(t)=$\frac{-3}{2}t^2$ d) x(t)=$3t^2$

9. The order and degree of D.E. $\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3 + y^2} = 0$ are respectively [GATE:2010] [ ]

a) 3 and 2 b) 2 and 3 c) 3 and 3 d) 3 and 1

10. The solution of $\frac{dy}{dx} = x^2y$ with the condition that y=1 at x=0 is [GATE:2007] [ ]

a)y=$e^{\frac{1}{2x}}$ b)lny=$\frac{x^3}{3} + 4$ c)lny=$\frac{x^2}{2}$ d)y=$e^{\frac{x^3}{3}}$

NUMERICAL METHODS & DIFFERENTIAL EQUATIONS

UNIT-V

HIGHER ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Objectives:

- To introduce the procedure for solving second and higher order differential equations with constant coefficients and its aplications in Engineering Problems.

Syllabus:

Solving Homogeneous differential equation, solving Non-Homogeneous differential equations when RHS terms are of the form e^{ax},sinax,cosax, polynomial in x , e^{ax} v(x) , x v(x)

Course Out comes: At the end of the UNITstudents will be able to

- Find general solution of both homogeneous and non-homogeneous equations
- Identify and apply initial and boundary conditions to find particular solutions to second and higher order homogeneous and non homogeneous differential equations manually and analyze and interpret the results.
- Solve applied problems encountered in engineering by formulating, analyzing differential equations of second and higher order.

Introduction:

Differential equations form the language in which the basic laws of physical science are expressed. The science tells us how a physical system changes from one instant to the next. The theory of differential equations then provides us with the tools and techniques to take this short term information and obtain the long-term overall behaviour of the system.

Definition:A D.E of the form $a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+...+a_{n-1}\frac{dy}{dx}+a_n y=Q(x)$ ______(1)

where a_0, a_1 ,..., a_n are constants and $Q(x)$ is a function of x is called a linear differential equation with constant coefficients of order n.

Definition: ***Homogeneous and non-homogeneous differential equations***

- If $Q(x)=0$ In equation(1) it is called homogeneous differential equation with constant coefficients.

  Example: $\frac{d^2 y}{dx^2}+2\frac{dy}{dx}+3y=0$,is a second order homogeneous differential equation.

- If $Q(x)\neq 0$ in equation (1), it is called non- homogeneous differential equation With constant coefficients.

  Example: $\frac{d^2 y}{dx^2}+2\frac{dy}{dx}+3y=\sin x$ is a second order non-homogeneous differential equation

.

Note:1)D≡ $\frac{d}{dx}$, **D²≡** $\frac{d^2}{dx^2}$ **,------**

Examples: $D\sin x=\frac{d}{dx}\sin x=\cos x$, $D^2\sin x=\frac{d^2}{dx^2}\sin x=-\sin x$

2) $\frac{1}{D} f(x) = \int f(x)dx$, $\frac{1}{D^2} f(x) = \iint f(x)dx\, dx$

Examples: $\frac{1}{D} x = \int x dx = \frac{x^2}{2}, \frac{1}{D^2}\sin x = \frac{1}{D}\left(\frac{1}{D}\sin x\right) = \frac{1}{D}\left(\int \sin x dx\right) = \frac{1}{D}(-\cos x) = -\int \cos x dx = -\sin x$

3)General solution of equation (1) = Complementary function + Particular integral

i.e., $y = y_c + y_p$

Working rule to find y_c : 1) write the given D.E in operator form as f(D)y = Q(x)

2) consider auxiliary equation f(m)=0 and find its roots

3) Depending upon the Nature of the roots we write y_c as follows:

NATURE OF ROOTS OF f(m)=0	y_c
1. $m_1, m_2, \ldots$ (Real and distinct roots)	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$ +----------
2. m_1, m_1, m_3----- (Two Real and equal roots)	$(c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x}$ +----------
3. a pair of imaginarc roots $m_1 = \alpha + i\beta$ $m_2 = \alpha - i\beta$	$(c_1 \cos\beta x + c_2 \sin\beta x)\, e^{\alpha x}$
4. $\alpha \pm i\beta, \alpha \pm i\beta, m_5$,---2 pairs of equal imaginary roots	$[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]\, e^{\alpha x} + c_5 e^{5x}$ +---------

<u>Note:</u> 1) To find y_p we have to consider

$$y_p = \frac{1}{f(D)} Q(x)$$

2)When Q(X) = 0 , $\mathbf{y_p} = 0$ i.e., in a homogeneous D.E always $\mathbf{y_p} = 0$

3)When Q(X)$\neq 0$ i.e.,in a non-homogeneous D.E following cases arises

$e^{ax}, e^{ax+b}, e^{ax-b}, a^x, k, cohax,$

$\sin ax,\ \cos ax,\ \sin(ax \pm b),\ \cos(ax \pm b)$ *a polynomial in x* , $e^{ax}v(x)$, $x^k v(x)$

<u>Working rule to find y_p under case(1):</u>

We know that

$$y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad , \quad \text{if } f(a) \neq 0$$

Example:

$y_p = \frac{1}{D^2+D+1}e^{-2x} = \frac{e^{-2x}}{3}$, since f(-2) $\neq 0$

Case1) :if f(a) = 0 , then

$y_p = \frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}$

Example: $y_p = \frac{1}{D^2+D}e^{-x} = \frac{x}{2D+1}e^{-x} = \frac{xe^{-x}}{-1}$, since f(-1) =0 and $f^1(-1) \neq 0$

Case2): if $f'(a) = 0$, then $y_p = \frac{1}{f(D)}e^{ax} = x^2 \frac{1}{f''(a)}e^{ax}$,if $f^{11}(a) \neq 0$ and so on

Example: $y_p = \frac{1}{(D+3)^2}e^{-3x} = \frac{x^2}{2}e^{-3x}$ ($\because$ $f^1(D)=2D+6 \Longrightarrow f^1(-3)= 0$ but $f''(D)=2 \Rightarrow f''(-3) \neq$
0)

Working rule to find y_p under case(2) :

We know that $y_p = \frac{1}{f(D)}\sin ax$, Let us consider $f(D)=\emptyset(D^2)$ Then $y_p = \frac{1}{\phi(D^2)}\sin ax$

Casei): Now replace $D^2 = -a^2$ if $\phi(-a^2) \neq 0$

Caseii): If $\phi(D^2) = \phi(-a^2)$ =0 then we proceed as shown in below examples(3) and (4)

Example : 1) $y_p = \frac{1}{D^2-4}\cos 2x = \frac{1}{-2^2-4}\cos 2x = \frac{-1}{8}\cos 2x$

Example : 2) $y_p = \frac{1}{D^3+4}\sin 2x = \frac{1}{(-2^2)D+4}\sin 2x = \frac{(4+4D)}{(4+4D)(4-4D)}\sin 2x$

$= \frac{(4+4D)}{16-16D^2}\sin 2x$

$= \frac{(1+D)}{4-4(-2^2)}\sin 2x$

$= \frac{(1+D)\sin 2x}{20}$

$= \frac{\sin 2x + 2\cos 2x}{20}$

Example : 3) $y_p = \frac{1}{D^2+3^2}\cos 3x = \frac{x}{2D}\cos 3x = \frac{x}{2}\int \cos 3x dx = \frac{x.\sin 3x}{2.3} = \frac{x\sin 3x}{6}$

Example : 4)y $_p$ =

$$\frac{1}{D^4-1}\sin x=\frac{x}{4D^3}\sin x=\frac{x}{4D.D^2}\sin x=\frac{x}{4D.-1^2}\sin x=\frac{x}{-4D}\sin x=\frac{-x}{4}\int\sin xdx=\frac{x\cos x}{4}$$

Note: Before finding y $_p$ under case(3) , remember the following expansions

I. $(1+D)^{-1} = 1-D+D^2-D^3+D^4-$...................................
II. $(1-D)^{-1} = 1+D+D^2+D^3+D^4-$...................................
III. $(1+D)^{-2} = 1-2D+3D^2-4D^3+5D^4-$...................................
IV. $(1-D)^{-2} = 1+2D+3D^2+4D^3+5D^4-$...................................
V. $(1+D)^{-3} = 1-3D+6D^2-10D^3+$...................................
VI. $(1-D)^{-3} = 1+3D+6D^2+10D^3+$...................................

Working rule to find y_p under case(3):

We know that y $_p$ = $\frac{1}{f(D)}Q(x)$,where $Q(x)$ is a polynomial in x

convert $\frac{1}{f(D)}$ into $(1+\psi)^{-1}$ where ψ is a function of D's ,then using above expansions we get y $_p$

Example : 1) Consider y $_p$ = $\frac{1}{D^2+3}x^2$

$$=\frac{1}{3}\frac{1}{(1+\frac{D^2}{3})}x^2$$

$$=\frac{1}{3}\left(1+\frac{D^2}{3}\right)^{-1}x^2$$

$$=\frac{1}{3}\left(1-\frac{D^2}{3}+\left(\frac{D^2}{3}\right)^2-....\right)x^2$$

$$=\frac{1}{3}\left(x^2-\frac{2}{3}\right)$$

Example : 2) $y_p = \frac{1}{D^3-4D}3x^2$

$$=\frac{1}{D(D^2-4)}3x^2$$

$$=\frac{3}{D}\frac{1}{-4(1-\frac{D^2}{4})}x^2$$

$$=\frac{-3}{4D}\left(1-\frac{D^2}{4}\right)^{-1}x^2$$

$$=\frac{-3}{4D}\left(1+\frac{D^2}{4}+\frac{D^4}{16}+---\right)x^2$$

$$=\frac{-3}{4D}\left(x^2+\frac{1}{2}\right)$$

$$= \frac{-3}{4}\left(\frac{x^2}{D} + \frac{1}{2D}\right)$$

$$= \frac{-3}{4}\left(\frac{x^3}{3} + \frac{x}{2}\right)$$

Working rule to find y_p under case(4):

We know that $y_p = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} e^{ax} v(x)$$

$$= e^{ax} \frac{1}{f(D+a)} v(x)$$

Depending on the nature of V(x) solve it further

Example: 1)y_p = $\frac{1}{D+2} e^{3x} x$

$$= e^{3x} \frac{1}{(D+3)+2} x$$

$$= e^{3x} \frac{1}{D+5} x$$

$$= e^{3x} \frac{1}{5(1+\frac{D}{5})} x$$

$$= e^{3x} \frac{1}{5}\left(1+\frac{D}{5}\right)^{-1} x$$

$$= e^{3x} \frac{1}{5}\left(1-\frac{D}{5}+\frac{D^2}{5^2}-.....\right) x$$

$$= \frac{e^{3x}}{5}\left(x-\frac{1}{5}\right)$$

Example : 2) $y_p = \frac{1}{D^2-6D+13} 8e^{3x} \sin 2x$

$$= 8e^{3x} \frac{1}{(D+3)^2-6(D+3)+13} \sin 2x$$

$$= 8e^{3x} \frac{1}{D^2+4} \sin 2x$$

$$=8e^{3x} . \frac{x}{2D} \sin 2x$$

$$=8e^{3x} .-\frac{x}{4} \cos 2x$$

$$= -2 x e^{3x} \cos 2x$$

Working rule to find y_p under case(5):

We know that $y_p = \frac{1}{f(D)} Q(x) = \frac{1}{f(D)} x^k v(x)$ **Note:** $e^{i\theta} = \cos\theta + i\sin\theta$

(arrows below the Note: $\cos\theta$ → R.P of $e^{i\theta}$; $\sin\theta$ → I.P of $e^{i\theta}$)

Case(1): Let k = 1 then $y_p = \left[x - \frac{f^1(D)}{f(D)} \right] \frac{1}{f(D)} v(x)$

Case(2): i) Let $k \neq 1$ and $v(x) = \sin ax$

$Y_p = \frac{1}{f(D)} x^k \sin ax$

$= \frac{1}{f(D)} x^k$ I.P of e^{iax}

= I.P of $\frac{1}{f(D)} x^k e^{iax}$

= I.P of $e^{iax} \frac{1}{f(D+ia)} x^k$

By using previous related method we will solve it

finally replace e^{iax} = cosax + i sinax

ii) Let $k \neq 1$ and $v(x) = \cos ax$

$Y_p = \frac{1}{f(D)} x^k \cos ax$

$= \frac{1}{f(D)} x^k$ R.P of e^{iax}

= R.P of $\frac{1}{f(D)} x^k e^{iax}$

= R.P of $e^{iax} \frac{1}{f(D+ia)} x^k$

By using previous related method we will solve it

finally replace e^{iax} = cosax + i sinax

Example: $Y_p = \frac{1}{D^2} x \sin 2x$

$= \frac{1}{D^2} x \, I.P of \; e^{i2x}$

= I.P of $\frac{1}{D^2} x e^{i2x}$

= I.P of $e^{i2x} \frac{1}{(D+2i)^2} x$

= I.P of $e^{i2x} \frac{1}{-4(1+\frac{D}{2i})^2} x$

= I.P of $\frac{-e^{i2x}}{4} . \left(1+\frac{D}{2i}\right)^{-2} x$

= I.P of $\frac{-e^{i2x}}{4} . \left(1 - 2\frac{D}{2i} + 3\frac{D^2}{(2i)^2} - - - -\right) x$

= I.P of $\frac{-e^{i2x}}{4} . \left(x - \frac{1}{i}\right)$

$$= \quad \text{I.P of } \frac{-e^{i2x}}{4}.(x+i)$$

$$= \quad \text{I.P of } \left(\frac{-\cos 2x - i\sin 2x}{4}\right)(x+i)$$

$$= \quad \frac{-\cos 2x}{4} - \frac{x\sin 2x}{4}$$

$$= -\frac{1}{4}(\cos 2x + x\sin 2x)$$

Assignment-Cum-Tutorial Questions

SECTION A

1. Solution of $(D^2 - a^2)y = 0$ is ____________________
2. The general solution of the D.E. $(D^4 - 6D^3 + 12D^2 - 8D)\, y = 0$ is ____________
3. Solution of $D^3 y = 0$ is ________________________
4. The particular integral of $(D^2 + 4^2)\, y = \sin 6x$ is ______________________
5. $\frac{1}{D^2} x^2 =$________________
6. $D^2(2x + 4) =$ _______________
7. The complete solution of the equation $f(D)\, y = Q(x)$ is ___________
8. Roots of Auxiallary equation $m^4 + 4 = 0$ are ___________
9. $\frac{1}{f(D^2)} \sin ax =$______________
10. The real and imaginary part of x^2e^{i3x} is _____ and ______ respectively
11. $\frac{1}{f(D)} e^{ax} v(x) =$______________
12. Roots of auxiliary equation $m^2(m^2 + 4) = 0$ are ________
13. Y_p of $\frac{1}{D^2 + 2D} e^{-2x}$ = _____________________________
14. In a homogenous linear D.E. $f(D)\, y = 0$, the general solution of y is ____________
15. In a non-homogenous linear D.E. $f(D)\, y = Q(x)$, then the general solution of y is _______
16. $\frac{1}{D - a} e^{ax}$ = _____________
17. $\frac{1}{D^2 - 5D} x$ = _______________
18. P.I. of $\frac{1}{f(D)} xv(x)$ = ______________________
19. P.I of $(D-1)^2 y = e^x \sin x$ is _______
20. The solution of the D.E $(D^2 - 2D + 5)^2 y = 0$ is________________
21. The solution of the differential equation $y'' + y = 0$ satisfying the conditions y(0)=1 and $y(\pi/2)$=2 is _________
22. The general solution of $(4D^3 + 4D^2 + D)y = 0$ is _________________
23. P.I. of $\frac{e^{-x}}{D^2 + D + 1}$ is _______________

Multiple Choice Questions:

1. Solution of $(D^3 + D)\, y = 0$ is [ ]
   a) $y = A\cos x + B\sin x$ b) $y = Ae^{x} + Be^{-x}$ c) $y = A + Be^{x} + Ce^{-x}$ d) $y = A + B\cos x + C\sin x$
2. Solution $(D^3 - D^2)y = 0$ is [ ]
   a) $y = Ae^{x} + B$ b) $y = (A + Bx)\, e^{x} + C$ c) $y = A + Bx + Ce^{x}$ d) none
3. P.I. of $\left(\frac{1}{D^2 + 1}\right) \cos^2 x =$ [ ]
   a) cosx b) – cosx c) sinx d) – sinx
4. General solution of $(D^2 - 1)y = x^2 + x$ is [ ]

a) $y = Ae^{x} + Be^{-x} + (x^2 + x + 2)$ b) $y = Ae^{x} + Be^{-x} - (x^2 + x + 2)$

c) $y = Ae^{x} + Be^{-x} + 1$ d) $y = A\cos x + B\sin x - 1$

5. P.I. of $(D + 1)^2 y = e^{-x} \cdot x$ is [ ]

a) $e^{-x} \cdot \frac{x^2}{2}$ b) $e^{-x} \cdot \frac{x^3}{6}$ c) $e^{-x} \cdot \frac{x^4}{24}$ d) $\frac{e^{-x}}{24}$

6.The complementary function of (D^3+D) y = 5 is ________ [ ]

a)a+bcosx+csinx b)bcosx+csinx c)a+bcosx d)none

7. C.F of $(D^2 + 4D + 13)y = e^{-2x}\sin 3x$ is __________ [ ]

a) $A\sin 3x + B\cos 3x$ b) $e^{-3x}(A\cos 2x + B\sin 2x)$ c) $e^{-2x}(A\cos 3x + B\sin 3x)$ d)none

8. $\frac{1}{(D-2)^3}e^{2x} =$ ________________ [ ]

a) $\frac{x^2e^{2x}}{6}$ b) $\frac{x^3e^{2x}}{6}$ c) $\frac{x^2e^{2x}}{4}$ d)none

9.The particular integral of $(D^2 - 4)y = \sin 3x$ is ____________[ ]

a) $\frac{1}{4}$ b) $\frac{-1}{13}$ c) $\frac{1}{5}$ d)None

10. $e^{-x}\left(a\cos\sqrt{3x} + b\sin\sqrt{3x}\right) + ce^{2x}$ is the general solution of [ ]

SECTION B:

1. a) $(D^3 + 4)y = 0$ b) $(D^3 - 8)y = 0$ c) $(D^3 + 8)y = 0$ d) $(D^3 - 2D^2 + D - 2)y = 0$
   Solve(D^2-4D+4)y=0
2. Obtain the general solution of (D-2)$(D+1)^2$y =0.
3. Give examples of C.F. for different nature of roots of an auxiliary equation
4. Find particular solution of initial value problem $y''+2y'+2y = 0$ with y(0) = 1 $y^1(0)$ = -1
5. It is given that $y''-2y'+y = 0$, with y(0) = 0 , y(1)=0 then what is y(1) ?
6. Given that $x''+3x = 0$ and x(0) = 1, x'(0) = 0 then what is x(1).
7. Solve $y''-y'-2y = 3e^{3x}$, y(0) = 0 , $y^1(0)$ = 2
8. Solve (D^3-5D^2+8D-4) y = $e^{2x.}$
9. Solve (D+2)$(D-1)^2$ y = 2 sinhx
10. Solve:(D^2-4D +3)y = sin 3xcos 2x
11. Solve (D^3+1) y = cos (2x-1)
12. Solve (D^2 - 1)y = 2e^x + 3x
13. Solve (D^4-4D+4) y = e^{2x} +x^2 + sin 3x.
14. Find y of $(D^3 - 7D^2 + 14D - 8)y = e^x\cos 2x$
15. Solve (D^2-2D+1) y = x e^x sinx.
16. Solve (D^2 – 4D +4) y = 8x^2e^{2x}sin2x .

GATE QUESTIONS

1.The solution for the differential equation $\frac{d^2x}{dt^2} = -9x$ with initial conditions x(0)=1 and $\frac{dx}{dt}$ at t =0 is 1 is__________________GATE(2014)

2.For the differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$ with initial conditions x(0) = 1 and $x^1(0)$= 0, the solution is________________ GATE(2010)

3. For the differential equation y''+2y'+y=0,y(0)=0,y(1)=0 the value of y(0.5) is ____________ GATE(2008)

4.If $y = f(x)$ is the solution of $\frac{d^2y}{dx^2}$ =0 with the bounndary conditiions y=5 at *x=0 and* $\frac{dy}{dx}$ =2 at $x = 10$, $f(15) =$ ________ GATE(2014)

5.The solution to the differential equation $\frac{d^2u}{dx^2} - k\frac{du}{dx}$ =0 where k is a constant, subjected to the boundary conditions u(0)=0 and u(L)=U, is________________ GATE(2008) The solution for the differential equation $\frac{d^2x}{dt^2} = -9x$ with initial conditions x(0)=1 and $\frac{dx}{dt}$ at t =0 is 1 is___________________GATE(2014)

NUMERICAL METHODS AND DIFFERENTIAL EQUATIONS

UNIT-VI
Partial differentiation

Course Objectives:

- To introduce the concept of total derivative, Jacobian & maxima and minima

Syllabus:

Total Derivative , chain Rule , Jacobian – Application – Finding Maxima and Minima of functions of two / three variables

Course Out comes: At the end of the course students will be able to

- Find total derivative of the given function
- Verify the functional dependence of functions
- Find maxima and minima of functions of two / three variables

Partial Differentiation:-

Let z = f (x,y) be a function of two variables x and y.

$Then \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$, if it exists is said to be partial derivative of z of f (x, y) w.r.t "x"; It is denoted by the symbol $\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x} or f_x$ i.e. The partial derivative of z = f (x,y) with respect to "x"is done, y is kept constant.

Similarly the partial derivative of z = f(x,y) wrt "y" keeping "x" constant is defined by

$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$ and it is denoted by $\frac{\partial z}{\partial y} or\ f_y$

In the same way, the partial derivatives of the function z = f(x_1, x_2………x_n) w.r.t "x_i" keeping other variables constant can be defined by

$$\frac{\partial z}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2....x_i + \Delta x_i....x_n) - f(x_1, x_2....x_i......x_i)}{\Delta x_i}, i = 1,2....n$$

Higher Order Partial Derivatives:-

In general the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of x and y and they can be differentiated repeatedly to get higher order partial derivatives.

So $\frac{\partial}{\partial x} = \left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, $\quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

$\frac{\partial}{\partial x} = \left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}$, $\quad \left(\frac{\partial}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

<u>Model Problems</u>:-

Total Derivative:

If u = f (x,y), where x = $\varphi(t)$, $y=\psi(t)$ then we express u as a function of t alone by substituting the values of x and y in f(x,y); thus we can find ordinary derivative $\frac{du}{dt}$ is called the <u>total derivative of u</u> to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$.

Chain Rule:

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\times\frac{dy}{dt}$$

$$=\frac{\partial u}{\partial x}\times\frac{dx}{dt}+\frac{\partial u}{\partial y}\times\frac{dy}{dt}------------------(1)$$

In three variables we get when u = f(x,y,z)

Where x, y,z are all functions of a variable t, then $\frac{du}{dt}=\frac{\partial u}{\partial x}\times\frac{dx}{dt}+\frac{\partial u}{\partial y}\times\frac{dy}{dt}+\frac{\partial u}{\partial z}\times\frac{dz}{dt}$

Differentiation of implicit functions:-

If f (x,y) = c be an implicit relation between x and y which defines as a differentiable function of x when t = x in (1), it becomes

In implicit function (2) becomes

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\times\frac{dy}{dx}------------------(2)$$

$$0=\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\times\frac{dy}{dx}$$

$$\therefore\frac{df}{dx}-\frac{\partial f}{\partial x}-\frac{\partial f/\partial x}{\partial f/\partial y}=\frac{dy}{dx}$$

1. Show that $\frac{\partial x}{\partial u}=\frac{1}{r}\frac{\partial y}{\partial\theta};\quad\frac{\partial y}{\partial u}=-\frac{1}{r}\frac{\partial x}{\partial\theta}$ and hence show that $\frac{\partial^2x}{\partial r^2}+\frac{1}{r}\frac{\partial x}{\partial r}+\frac{1}{r^2}\frac{\partial^2x}{\partial\theta^2}=0$

If $x=e^{r\cos\theta}.cis(r\sin\theta)$

$y=e^{r\cos\theta}.\sin(r\sin\theta)$

Sol: x = $e^{r\cos\theta}\cos(r\sin\theta)$

$$\frac{\partial x}{\partial r}=e^{r\cos\theta}[-\sin(r\sin\theta)\times\sin\theta]+[\cos(r\sin\theta)]\times e^{r\cos\theta}\times\cos\theta$$

$$=e^{r\cos\theta}[-\sin\theta\sin(r\sin\theta)]+\cos\theta\cos(r\sin\theta).$$

$$=e^{r\cos\theta}[\cos\{\theta+r\sin\theta\}]------------------(1)$$

$$y=e^{r\cos\theta}\sin(r\sin\theta)$$

$$\frac{\partial y}{\partial r}=e^{r\cos\theta}\times\cos(r\sin\theta)\times\sin\theta+\sin(r\sin\theta)e^{e\cos\theta}\cos\theta$$

$$=e^{r\sin\theta}[\sin\theta\times\cos(r\sin\theta)+\cos\theta\times\sin(r\sin\theta)]$$

$$=e^{r\sin\theta}[\sin(\theta+r\sin\theta)]------------------(2)$$

$$\frac{\partial x}{\partial \theta} = e^{r\cos\theta}\left[-\sin(r\sin\theta)\times r\cos\theta\right] + \cos(r\sin\theta)\times e^{r\cos\theta}\times(-r\sin\theta)$$

$$= -re^{r\cos\theta}\left[+\cos\theta\sin(r\sin\theta) + \sin\theta\cos(r\sin\theta)\right]$$

$$= re^{r\cos\theta}\left[\sin(\theta + r\sin\theta)\right] ----------------(3)$$

$$\frac{\partial y}{\partial \theta} = e^{r\cos\theta}\left[\cos(r\sin\theta)\times r\cos\theta + \sin(r\sin\theta)e^{r\cos\theta}\times(-e\sin\theta)\right]$$

$$= re^{r\cos\theta}\left[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)\right]$$

$$= re^{r\cos\theta}\cos(\theta + r\sin\theta) ----------------(4)$$

To show that $\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta}$

$$e^{r\cos\theta}\cos(\theta + r\sin\theta)$$

$$= \frac{1}{r}\times\left[re^{r\cos\theta}\cos(\theta + r\sin\theta)\right] \text{ equal}$$

To show that $\frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}$

$$e^{r\cos\theta}\sin\left[\theta + r\sin\theta\right]$$

$$= -\frac{1}{r}\left[-r^{r\cos\theta}e\sin(\theta + r\sin\theta)\right]$$

$$= e^{r\cos\theta}\sin(\theta + r\sin\theta)$$

Simple Method:-

$$\frac{\partial^2 x}{\partial x^2} = \frac{\partial}{\partial u}\left(\frac{1}{r}\frac{\partial y}{\partial \theta}\right) = \frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta} + \left(\frac{\partial y}{\partial \theta}\right)\left(-\frac{1}{r^2}\right)$$

$$\frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = \frac{1}{r^2}\left[\frac{\partial}{\partial \theta}\left(-r\frac{\partial y}{\partial r}\right)\right] = -r\frac{\partial^2 y}{\partial \theta 0 r}$$

$$\frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial x}{\partial r} - r\frac{\partial^2 y}{\partial x\partial \theta}\times\frac{1}{r^2}$$

$$\frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial x}{\partial r} - \frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta}$$

$$-\frac{1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\times\frac{1}{r}\frac{\partial y}{\partial \theta} = 0$$

Jacobians:-

If u and v are functions of two independent variables x and y, then the determinant $\begin{bmatrix}\partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial x/\partial y\end{bmatrix}$ is called the Jacobian of u, v with respect to x,y and is written as $\frac{\partial(u,v)}{\partial(x,y)} or J\left(\frac{u,v}{x,y}\right)$

Similarly the Jacobian of u, v, w with respect to x,y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix}\partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z\end{vmatrix} = \begin{bmatrix}u_x & u_y & u_z \\ v_x & v_y & u_z \\ w_x & w_y & w_z\end{bmatrix}$$

Properties of Jacobians:-

(1) If $J=\frac{\partial(u,v)}{\partial(x,y)}\, and\ J^1=\frac{\partial(x,y)}{\partial(u,v)}\ \ then \qquad S.T\ JJ^1=1$

Proof: Let u = f (x,y) and v = g (x,y)

After solving for x and y, suppose we have $x=\varphi(u,v)\ and\ y=\psi(u,v)$ thus

$$\frac{\partial u}{\partial u}=1=\frac{\partial u}{\partial x}\times\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\times\frac{\partial y}{\partial u}$$

$$\frac{\partial u}{\partial v}=0=\frac{\partial u}{\partial x}\times\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\times\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial u}=0=\frac{\partial v}{\partial x}\times\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}\times\frac{\partial y}{\partial u}$$

$$\frac{\partial v}{\partial v}=1=\frac{\partial u}{\partial x}\times\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\times\frac{\partial y}{\partial v}$$

$$\therefore\ \frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial x}{\partial y}\end{vmatrix}\times\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial x}{\partial y}\end{bmatrix}\times\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial v}{\partial v} & \frac{\partial y}{\partial v}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=1$$

Property:-

If u, v are functions of r,s and r,s are functions of x,y then S.T. $\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\times\frac{\partial(r,s)}{\partial(x,y)}$

Sol: Consider RHS

$$\frac{\partial(u,v)}{\partial(r,s)}\times\frac{\partial(r,s)}{\partial(x,y)}=\begin{bmatrix}\partial u/\partial r & \partial u/\partial s\\ \partial v/\partial r & \partial v/\partial s\end{bmatrix}\times\begin{bmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{\partial u}{\partial r}\times\frac{\partial r}{\partial s}+\frac{\partial u}{\partial s}\times\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\times\frac{\partial s}{\partial y}\\ \frac{\partial v}{\partial r}\times\frac{\partial u}{\partial x}+\frac{\partial v}{\partial s}\times\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y}+\frac{\partial v}{\partial s}\times\frac{\partial s}{\partial y}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}=\frac{\partial(u,v)}{\partial(x,y)}=LHS$$

Model Problem:

Sol: $u = \frac{yz}{x}$ $\quad \frac{\partial u}{\partial x} = u_x = -\frac{yz}{x^2}$ $\quad \frac{\partial u}{\partial y} = \frac{u}{y} = z/x$ $\quad \frac{\partial u}{\partial z} = u_z = y/x$

$v = \frac{xz}{y}$ $\quad \frac{\partial v}{\partial x} = z/y$ $\quad \frac{\partial v}{\partial y} = -\frac{xz}{y^2}$ $\quad \frac{\partial v}{\partial z} = x/y$

$w = \frac{xy}{z}$ $\quad \frac{\partial w}{\partial x} = \frac{y}{z}$ $\quad \frac{\partial w}{\partial y} = \frac{x}{z}$ $\quad \frac{\partial w}{\partial z} = \frac{-xy}{z^2}$

$$\partial\left(\frac{u,v,w}{x,y,z}\right) = \begin{bmatrix} -yz/x^2 & z/x & y/x \\ z/x & -xz/y^2 & x/y \\ y/x & x/y & -xy/z^2 \end{bmatrix}$$

Multiply C_1 with x

C_2 with y

C_3 with z

$$= \begin{bmatrix} -yz/x^2 & z/x & y/x \\ z/x & -xz/y^2 & x/y \\ y/x & x/y & -xy/z^2 \end{bmatrix} \Rightarrow \frac{1}{y_z}\begin{bmatrix} -xyz/x^2 & yz/x & yz/x \\ xz/y & -xyz/y^2 & xz/y \\ xy/x & yx/y & -xyz/z^2 \end{bmatrix}$$

$$= \frac{1}{xyz} \times \frac{yz}{x} \times \frac{xz}{y} \times \frac{xy}{z} \times \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Taking common $\frac{yz}{x}$ from R_1

$\frac{xz}{y}$ from R_2

$\frac{xy}{z}$ from R_3

$$= \frac{x^2y^2z^2}{(xyz)^2}\left[-1(1-1)-1(-1-1)+1(1+1)\right]$$

= -1 (-2) +1 x 2 = 4

Functional Dependence: -

If u = f (x,y) and v = g (x,y) are two given differentiable functions in the dependent variables x,y; suppose these functions are connected by a relation F (u, v) = 0 where F is differentiable.

We say that u and v functionally dependent on one another , if the partial derivatives u_x, u_y, v_x, v_y are all not zero at a time.

Theorem:-

If the functions u and v of the independent variable x and y are functionally dependent then the Jacobian vanishes.

Proof:-Consider F (u, v) = 0

Differentiating F (u, v) = 0 partially wrt "x and y, we get $\frac{\partial F}{\partial u}\times\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\times\frac{\partial v}{\partial x}=0$

$$\frac{\partial F}{\partial u}\times\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\times\frac{\partial v}{\partial y}=0$$

A Non-trivial solution $F_u \neq 0$; $F_v \neq 0$, to this system exists if the coefficient determinant is zero.

$$\Rightarrow \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0 \text{ i.e. } \frac{\partial(u,v)}{\partial(x,y)}=0$$

Note:- If the Jacobian $J\left(\frac{u,v}{x,y}\right)=0$ then u and v are said to be functionally independent.

Model Problems:

Show that the functions u = xy + yz + zx,v = $x^2 + y^2 + z^2$ and w = x + y + z are functionally related. Find the relation between them?

Sol:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} = \begin{bmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{bmatrix}$$

u = xy + yz + zx	u_x = y + z	u_y = x +z	u_z = y + x
v = $x^2 + y^2 + z^2$	v = 2x	v_y = 2y	w_x = 1
w = x - y + z	w_x = 1	w_y = 1	w_z = 1

$R_1 + R_2$

$$= 2\times\begin{bmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{bmatrix} = 2(x+y+z)\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{pmatrix}$$

$$= 2\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ 0 & 0 & 0 \end{bmatrix}_{R_3-R_1} = 0$$

u,v,w are functionally dependent $\Rightarrow$ Functional relationship exists among u,v,w.

Now $w^2 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

$= v + 2u$

$\therefore w^2 = v + 2u$

Maxima and Minima values of f (x,y)

Working Rule to find the Maximum and Minimum values of f (x,y):-

- Find $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ and equate each to zero. Solve these as simultaneous equations in x and y. Let (a,b) (c,d) be the pairs of values.
- Calculate the value of $r=\frac{\partial^2 f}{\partial x^2}$, $s=\frac{\partial^2 f}{\partial x\partial y}$, $t=\frac{\partial^2 f}{\partial y^2}$ for each pair of values.
- (i) If $rt - s^2 > 0$ and $r < 0$ at (a,b), f (a,b) is a Max. value
  (ii) If $rt - s^2 > 0$ at (a,b), f (a,b) is a Mini value
  (iii) If $rt - s^2 < 0$ at (a,b), f (a,b) is not an extreme value. i.e. (a,b) is a saddle point.
  (iv) If $rt - s^2 = 0$ at (a,b), the case is doubtful and needs further investigation.

Model Problems:-

Examine the following function for extreme values?

Sol: $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$f_x = 4x^3 - 4x + 4y$ $\qquad$ $f_y = 4y^3 + 4x - 4y$

$f_{xx} = 12x^2 - 4 = r$ $\qquad$ $f_{yy} = t = 12y^2 - 4$

$f_{xy} = s = 4$

Now If $f_x = 0$ $\qquad$ if $f_y = 0$

$x^3 - x + y = 0$ $\qquad$ $y^3 - y + x = 0$

$y^3 + x - y = 0$

$x^3 + y^3 = 0 \Rightarrow (x+y)\left[x^2 - xy + y^2\right] = 0$

$\Rightarrow x = -y$

Putting $x = -y$ in $f_x = 0 \Rightarrow x^3 - x - x = 0$

$x^3 - 2x = 0$

$x^2 - 2 = 0 \Rightarrow x^2 = 2$

$x = \pm\sqrt{2}$

$y = \mp\sqrt{2}$

(i) At $\left(\sqrt{2}, -\sqrt{2}\right)$, $rt - s^2 = \left[12\left(\sqrt{2}\right)^2 - 4\right]\left[12\times 2 - 4\right] - 4^2$

$= 20 \times 20 - 4^2 = 400 - 16 = 384 > 0.$

Hence $f\left(\sqrt{2}, -\sqrt{2}\right)$ is a min value.

At $\left(\sqrt{2}, -\sqrt{2}\right) \Rightarrow rt - s^2 = \left[12\left(-\sqrt{2}\right)^2 - 4\right]\left[12\left(\sqrt{2}\right)^2 - 4^2\right]0 \text{ and } r = 12\left(-\sqrt{2}\right)^2 - 4 > 0$

Hence $f\left(-\sqrt{2}, \sqrt{2}\right)$ is also a min value.

(ii) At (0,0) $rt - s^2 = [12 \times 0^2 - 4]\,[12 \times 0^2 - 4] - 4^2$

$= (-4)(-4) - 4^2 = 0$

$\therefore$ Further investigation is needed.

(iii) Now $f(0,0) = 0$ and for points along the x – Axis where $y = 0$, $f(x,y) = x^4 - 2x^2 = x^2(x^2 - 2)$ which is negative for points in the neighborhood of the origin.

Thus in the neighborhood of (0,0) there are points

When $f(x,y) < f(0,0)$ and there are points where $f(x,y) > f(0,0)$

Hence f (0,0) is not Extreme Value i.e. it is a saddle point.

Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube?

Sol:

Let 2x, 2y, 2z be the length, breadth and height of the rectangular solid so that its volume $V = 8xyz$

Let R be the redius of the sphere so that $x^2 + y^2 + z^2 = R^2$

Then $F(x,y,z) = 8xyz + \lambda\left[x^2 + y^2 + z^2 - R^2\right]$ and $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0; \frac{\partial F}{\partial z} = 0$ given

$8yz + 2\lambda x = 0$; $\quad 8xz + 2\lambda y = 0$; $\quad 8xy + 2\lambda z = 0$

$2\lambda x = -8yz$ or $2x^2\lambda = -8xyz = 2y^2\lambda = 2z^2\lambda$

$\Rightarrow 2x^2\lambda = 2y^2\lambda = 2z^2\lambda$

$x^2 = y^2 = z^2 \Rightarrow x = y = z$

$\therefore$ The Rectangular solid in a cube.

Assignment-Cum-Tutorial Questions

Section-A

Objective Questions:

1. If u = xlog(xy) where $x^3 + y^3 + 3xy = 1$ find $\frac{du}{dx}$.
2. If z = f (x,y), then write $\frac{\partial z}{\partial x}$?
3. If $u = e^{xyz}$, write the values of u_z, u_x,u_y
4. If r = x /y, s = y/z, t = z/x write the value of u_x, u_y, u_z
5. If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial r}{\partial x}, \frac{\partial x}{\partial \theta}$.
6. Explain Jacobian?
7. What is the value of J J^1=?
8. Explain extreme value?
9. Write the values of l,m,n value when f (x,y) = 0 in the sense of maximum and minimum?
10. Total derivative of u(x, y) is du = [ ]

    a) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ b) $\frac{\partial u}{\partial x}\,.\,dx + \frac{\partial u}{\partial y}\,.\,dy$ c) $\frac{\partial u}{\partial x}\,dx - \frac{\partial u}{\partial y}\,.\,dy$ d) $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}$
11. J . J1 = ---------- [ ]

    a) 1 b) Zero c) -1 d) none
12. If u = sin (x+y) then $\frac{\partial u}{\partial y}$ =---------- [ ]

    a) sin x b) cos (x+ y) c) tan (x+ y) d) none
13. If u = $J\left(\frac{u,v}{x,y}\right)$ then $J\left(\frac{x,y}{u,v}\right)$ = [ ]

    a) u b) 1/u c) 1 d) none
14. The minimum value of x2 + y2 + z2 given that x+y+z = 3a is [ ]

    a) 3a b) $4a^2$ c) $\frac{a^2}{3}$ d) $3a^2$
15. The stationary points of x3 y2 (1-x–y) are [ ]

    a) (0,1) b) (-1,-1) c) (1/2, 1/3) d) (1,1)
16. If the functions u & v of the independent variables x & y are functionally dependent then [ ]

    a) J = 0 b) $J \neq 0$ c) J = 1 d) $J \neq 1$
17. If ln – m2 > 0 & l < 0 then f(x, y) has [ ]

    a) minim mum value b) maximum value

    c) zero value d) neither maximum nor minimum
18. If f (x, y) = x2 + y2 + 6x+ 12 then minimum value f(x, y) is [ ]

    a) -3 b) 3 c) 0 d) none

19. If fx (a, b) = 0 , fy(a , b) = 0 then (a, b) is said to be [ ]
a) saddle point b) stationary point c) minimum point d) maximum point

Section-B

Subjective Questions

1. If $r^2 = x^2 + y^2 + z^2$ and u = r^m then Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ = m(m+1)r^{m-2}
2. If f (x,y) = Tan^{-1} (x + 2y) ,Find f_x, f_y
3. If f (u,v,t) = e^{uv} sin ut, Find f_u, f_v, f_t
4. If z is a function of x and y, where $x^2 + y^2 + z^2 = 1$ Find $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$
5. If f (x,y) = cos 3x X sin 4y find $f_x\left(\frac{\pi}{12},\frac{\pi}{6}\right)$ and $f_y\left(\frac{\pi}{12},\frac{\pi}{6}\right)$
6. For f (x,y) = x^7 log y + sin xy, Verify $f_{xy} = f_{yx}$
7. If u = $x^2 - 2y^2 + z^2 + z^3$, x sin , y = e^t, z = 3t find $\frac{du}{dt}$
8. If z = $u^3 v^5$, where u = x + y, v = x- y find $\frac{\partial z}{\partial y}$ by the chain rule.
9. If f (u,v,w) is differentiable, and u = x – y , v = y –z and w = z-x show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 6.$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

10. If x+y+z=u, y+z=uv, z=uvw, then evaluate
11. If u = x2 $Tan\frac{y}{x} - y^2 Tan^{-1}\frac{x}{y}$ show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$
12. If $\theta = t^n e^{-r^2/4t}$ what value of n will make $\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right)\right] = \frac{\partial \theta}{\partial t}$
13. Given that u = $e^{r\cos\theta}\cos(r\sin\theta)$
14. V = $e^{r\cos\theta}\sin(r\sin\theta)$
15. Prove tht $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}; \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$
16. If f (x,y) = (1-2xy + y2)-1/2 show that $\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial f}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2\frac{\partial f}{\partial y}\right] = 0$
17. u = f (r); x = r cos θ; y = r sin θ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$
18. If u = $\frac{yz}{x}; v = \frac{xz}{y}, w = \frac{xy}{z}$ show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

19. Show that the function u = x + y +z , v = x2 + y2 + z2 – 2xy – 2yz – 2zx and w = x3 + y3 + z3 – 3xyz are functionally related?
20. Find the max and min values of x3 + 3xy2 – 15x2 – 15y2 + 72x?
21. Find the Max and min values of $xy + \frac{e^3}{x} + \frac{e^3}{y}$
22. Find there positive numbers whose sum is k and whose product is maximum?
23. Find the min value of x2 + y2 + z2 where ax + by + cz = p.
24. A rectangular box open at the top has a capacity of 32 cubic feet. Find the dimensions of the box requiring least material for its construction.

GATE QUESTIONS

1. The minimum value of the function f(x)= $x^3-3x^2-24x+100$ in the interval[-3, 3] is--- GATE(2014)

   a)20 b) 28 c)16 d)32

2. The function f(x) = 2x−x2+3fx=2x-x2+3 has------- GATE(2011)

   a) a maxima at x = 1 and minimum at x = 5
   b) a maxima at x = 1 and minimum at x =− 5
   c) only maxima at x = 1 and
   d) only a minimum at x = 5 GATE(2014)

3. Let f(x)=xe−x. The maximum value of the function in the interval (0,∞) is------

   a) e^{-1} b) e c) 1- e^{-}1 d) 1+ e^{-}

4. The maximum value of $f(x) = x^3 - 9x^2 + 24x + 5$ in the interval [1, 6] is GATE-(2011)

   a) 21 b) 25 c) 41 d) 46